12th Philippine Mathematical Olympiad

Qualifying Stage for Regions 1, 2, and CAR

7 November 2009

Part I. Each correct answer is worth two points.

1. If $x \in \mathbb{R}$ and $x^2 - 2x - 2 \geq 0$, what is the least value of x^2?

(a) $1 - \sqrt{3}$ (b) $1 + \sqrt{3}$ (c) $4 - 2\sqrt{3}$ (d) $4 + 2\sqrt{3}$

2. How many points (x, y) in the xy-plane, with positive rational coordinates, satisfy the inequality $6x + 5y \leq 30$?

(a) 8 (b) 9 (c) 10 (d) infinite

3. Find all values of x that satisfy the equation $\log_3 2\sqrt{2x - 3} > 0$.

(a) $x > 13/8$ (b) $x > 3/2$ (c) $x > 5/2$ (d) $x > 7/8$

4. Let V be one of the vertices of a fair die. If the die is tossed on a table, what is the probability that vertex V will be in contact with the table?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

5. For what values of a does equation $(x - a)^2 + (x^2 - 3x + 2)^2 = 0$ have a real solution?

(a) 0 (b) 1 (c) 1 or 2 (d) 0, 1, or 2

6. Let x and y satisfy the equation $2y^2 - 2\sqrt{2}y - x - 6 = 0$. For what real values of x is y real?

(a) none (b) $x \geq -7$ (c) $x \leq -6$ (d) all reals

7. In how many ways can the letters of the word OLYMPIAD be rearranged in such a way that the vowels appear in alphabetical order and the consonants also appear in alphabetical order?

(a) $3! \cdot 5!$ (b) 816 (c) $\binom{6}{3}$ (d) 56

8. How many real values of x does the equation $\sqrt{x + 1} = x - 1$ have?

(a) zero (b) one (c) two (d) three

9. If $f(x) = 2 \cdot 3^x$, which of the following is equal to $f(1+x) - f(x)$?

(a) $2f(x)$ (b) $(1+x)f(x)$ (c) $-3f(x)$ (d) $f(1)$

10. If $3^x = 2^z$ and $3^y = 2^x$, which of the following inequalities is true?

(a) $x < y < z$ (b) $x < z < y$ (c) $y \leq z < x$ (d) $y < x \leq z$

11. How many integral values of n will make $35 - 12n + n^2$ a prime?

(a) none (b) one (c) two (d) infinite

12. A real number x satisfies the inequality $-y < x < y$ for all positive real numbers y. Which of the following is true?

(a) $x < \frac{1}{2}$ (b) $\frac{1}{2} \leq x < 1$ (c) $x = 1$ (d) $x > 1$

13. How many real roots does the equation $\Big|\big||x| + 3\big| - 2\Big| = 0$ have?

(a) none (b) one (c) two (d) three

14. The time required for each of the two examinees to solve any problem differs by 2 minutes. Together, they can solve 32 problems in an hour. How many minutes will it take the slower examinee to solve a problem?

(a) 3 (b) 4 (c) 5 (d) 6

15. The smallest and the largest interior angles of a convex polygon measure 90° and 180°, respectively. If the measures of its interior angles are in arithmetic progression, how many sides does the polygon have?

(a) 6 (b) 7 (c) 8 (d) 9

Part II. Each correct answer is worth three points.

16. Three lines parallel to a base of a triangle divide the other sides into four congruent segments. As a consequence, the area of the triangle is divided into four parts with unequal areas. If the area of the largest part is 28 cm^2, what is the area (in cm^2) of the original triangle?

(a) 52.5 (b) 56 (c) 64 (d) 98

17. Let $T = 3 + 6 + 9 + \cdots + 3N$, where N is the least integer such that $T \geq 2009$. What is $1 + 2 + 3 + \cdots + N$?

(a) 600 (b) 703 (c) 741 (d) 2109

18. A square is inscribed inside an equilateral triangle, where two vertices of the square are on one side of the triangle. Knowing that one side of the triangle is 1 cm long, how long (in cm) is one side of the square?

(a) $\frac{1}{2}$ (b) $2\sqrt{3}-3$ (c) $\frac{1}{3}$ (d) $\sqrt{3}-1$

19. Three of the roots of the equation $2x^4+ax^2+bx+c=0$ are -1, 2, and -3. What is b?

(a) 0 (b) 6 (c) 8 (d) -18

20. Find the area (in square units) of the closed region in the xy-plane that is bounded by the x-axis, the line $x=6$, and the graph of the function f defined by

$$y=f(x)=\begin{cases}2x & \text{if } 0\le x\le 2\\ -\frac{1}{2}x+5 & \text{if } 2\le x\le 6.\end{cases}$$

(a) 4 (b) 8 (c) 16 (d) 32

21. Circles O_1 and O_2 are externally tangent at point P, and have radii 2 cm and 4 cm, respectively. Points A and B, both different from P, are chosen on O_1 and O_2, respectively, such that A, P, and B are collinear. If $AB=4$ cm, how long is PB?

(a) 2 cm (b) $1+\sqrt{3}$ cm (c) $\frac{10}{3}$ cm (d) $\frac{8}{3}$ cm

22. What is the remainder when x^{2009} is divided by x^2-1?

(a) 1 (b) x (c) $x+1$ (d) $x-1$

23. One side of square $ABCD$ is 3 cm long. Let E and F be two points inside $ABCD$ such that $EF=1$ cm, $EF\|AB$, and A is nearer to E than F. Find the area of the non-convex hexagon $ABFCDE$.

(a) 5 cm^2 (b) 5.5 cm^2 (c) 6 cm^2 (d) 6.5 cm^2

24. For how many positive integers a less than 100 does the system

$$\begin{cases}x^2=y+a\\ y^2=x+a\end{cases}$$

have integral solutions (x,y)?

(a) 19 (b) 20 (c) 9 (d) 10

25. Let O be the center of a circle, and let BA and BC be the tangents from a point B to the circle at A and C, respectively. The smaller (circular) sector AOC is cut to form a right-circular cone. If $OB = 4$ cm and $\angle ABC = 60°$, determine the base radius of the cone.

(a) $\frac{2}{3}$ cm (b) $\frac{\sqrt{3}}{2}$ cm (c) $\sqrt{2}$ cm (d) 2 cm

Part III. Each correct answer is worth six points.

26. In $\triangle ABC$, let P and Q be the midpoints of sides AB and AC, respectively. Let BQ and CP intersect at the point G. If the area of $\triangle AGP$ is 3 square inches, what is the area of $\triangle ABC$ in square inches?

(a) 24 (b) 18 (c) $9\sqrt{2}$ (d) $15\sqrt{3}$

27. A positive integer is called *triangular* if it is equal to $\frac{1}{2}n(n+1)$ for some integer n. How many pairs (a, b) of triangular numbers are there such that $b - a = 2009$?

(a) four (b) five (c) six (d) seven

28. On an island, there are only two kinds of people: those who are always honest, and those who always lie. Three inhabitants are talking. (1) Andrew says "Bernie is honest." (2) Bernie says "Andrew and Cholo are both honest." (3) Cholo says "Andrew is a liar." Which of the following conclusions is correct?

(a) All three inhabitants are honest.
(b) Andrew and Bernie are honest, and Cholo is a liar.
(c) Andrew is honest, and Bernie and Cholo are liars.
(d) Andrew and Bernie are liars, and Cholo is honest.

29. Fifty distinct numbers, whose sum is 3000, are chosen at random from the set $\{1, 2, 3, \ldots, 100\}$. What is the least number of even numbers among these fifty numbers?

(a) 3 (b) 4 (c) 5 (d) 6

30. Simplify the following expression:

$$\frac{\sqrt{10+\sqrt{1}}+\sqrt{10+\sqrt{2}}+\sqrt{10+\sqrt{3}}+\cdots+\sqrt{10+\sqrt{99}}}{\sqrt{10-\sqrt{1}}+\sqrt{10-\sqrt{2}}+\sqrt{10-\sqrt{3}}+\cdots+\sqrt{10-\sqrt{99}}}.$$

(a) $1+\sqrt{2}$ (b) $\sqrt{20}$ (c) $2\sqrt{10}$ (d) $2+\sqrt{10}$

Answers

1. c	**6.** b	**11.** c	**16.** c	**21.** d	**26.** b
2. d	**7.** d	**12.** a	**17.** b	**22.** b	**27.** c
3. a	**8.** b	**13.** a	**18.** b	**23.** c	**28.** d
4. c	**9.** a	**14.** c	**19.** c	**24.** a	**29.** d
5. c	**10.** d	**15.** c	**20.** c	**25.** a	**30.** a