12^{th} Philippine Mathematical Olympiad

Area Stage

21 November 2009

Part I. No solution is needed. All answers must be in simplest form. Each correct answer merits two points.

1. If $a = 2^{-1}$ and $b = \frac{2}{3}$, what is the value of $(a^{-1} + b^{-1})^{-2}$?

2. Find the sum of all (numerical) coefficients in the expansion of $(x + y + z)^3$.

3. A circle has radius 4 units, and a point P is situated outside the circle. A line through P intersects the circle at points A and B. If $PA = 4$ units and $PB = 6$ units, how far is P from the center of the circle?

4. Let $y = (1 + e^x)(e^x - 6)^{-1}$. If the values of x run through all real numbers, determine the values of y.

5. The sum of the product and the sum of two integers is 95. The difference between the product and the sum of these integers is 59. Find the integers.

6. Let A, B, C, D (written in the order from left to right) be four equally-spaced collinear points. Let ω and ω' be the circles with diameters AD and BD, respectively. A line through A that is tangent to ω' intersects ω again at point E. If $AB = 2\sqrt{3}$ cm, what is AE?

7. A certain high school offers its students the choice of two sports: football and basketball. One fifth of the footballers also play basketball, and one seventh of the basketball players also play football. There are 110 students who practice exactly one of the sports. How many of them practice both sports?

8. Simplify: $\sqrt{\sin^4 15^\circ + 4\cos^2 15^\circ} - \sqrt{\cos^4 15^\circ + 4\sin^2 15^\circ}$.

9. Let a, b, and c be the roots of the equation $2x^3 - x^2 + x + 3 = 0$. Find the value of

$$\frac{a^3 - b^3}{a - b} + \frac{b^3 - c^3}{b - c} + \frac{c^3 - a^3}{c - a}.$$

10. In $\triangle ABC$, let D, E, and F be points on sides BC, CA, and AB, respectively, so that the segments AD, BE, and CF are concurrent at point P. If $AF : FB = 4 : 5$ and the ratio of the area of $\triangle APB$ to that of $\triangle APC$ is $1 : 2$, determine $AE : AC$.

11. A circle of radius 2 cm is inscribed in $\triangle ABC$. Let D and E be the points of tangency of the circle with the sides AC and AB, respectively. If $\angle BAC = 45^\circ$, find the length of the minor arc DE.

12. Two regular polygons with the same number of sides have sides 48 cm and 55 cm in length. What is the length of one side of another regular polygon with the same number of sides whose area is equal to the sum of the areas of the given polygons?

13. The perimeter of a right triangle is 90 cm. The squares of the lengths of its sides sum up to 3362 cm^2. What is the area of the triangle?

14. Determine all real solutions (x, y, z) of the following system of equations:

$$\begin{cases} x^2 - y = z^2 \\ y^2 - z = x^2 \\ z^2 - x = y^2. \end{cases}$$

15. For what value(s) of k will the lines $2x + 7y = 14$ and $kx - y = k + 1$ intersect in the first quadrant?

16. For what real numbers r does the system of equations

$$\begin{cases} x^2 = y^2 \\ (x-r)^2 + y^2 = 1 \end{cases}$$

have no solutions?

17. Determine the smallest positive integer n such that n is divisible by 20, n^2 is a perfect cube, and n^3 is a perfect square.

18. Find all pairs (a, b) of integers such that $\sqrt{2010 + 2\sqrt{2009}}$ is a solution of the quadratic equation $x^2 + ax + b = 0$.

19. Determine all functions $f : (0, +\infty) \to \mathbb{R}$ such that $f(2009) = 1$ and

$$f(x)f(y) + f\left(\tfrac{2009}{x}\right) f\left(\tfrac{2009}{y}\right) = 2f(xy)$$

for all positive real numbers x and y.

20. Find all pairs (k, r), where k is an integer and r is a rational number, such that the equation $r(5k - 7r) = 3$ is satisfied.

Part II. Show the solution to each problem. A complete and correct solution merits ten points.

21. Each of the integers $1, 2, 3, \ldots, 9$ is assigned to each vertex of a regular 9-sided polygon (that is, every vertex receives exactly one integer from $\{1, 2, \ldots, 9\}$, and two vertices receive different integers) so that the sum of the integers assigned to any three consecutive vertices does not exceed some positive integer n. What is the least possible value of n for which this assignment can be done?

22. Let E and F be points on the sides AB and AD of a convex quadrilateral $ABCD$ such that EF is parallel to the diagonal BD. Let the segments CE and CF intersect BD at points G and H, respectively. Prove that if the quadrilateral $AGCH$ is a parallelogram, then so is $ABCD$.

23. Let p be a prime number. Let a, b, and c be integers that are divisible by p such that the equation $x^3 + ax^2 + bx + c = 0$ has at least two different integer roots. Prove that c is divisible by p^3.

- end of the problems -

Answer Key

1. $\frac{4}{49}$

2. 3^3

3. $2\sqrt{10}$ units

4. $(-\infty, -\frac{1}{6}) \cup (1, +\infty)$

5. 11 and 7

6. 9 cm

7. 11 students

8. $\frac{1}{2}\sqrt{3}$

9. -1

10. $2:7$

11. $\frac{3}{2}\pi$ cm

12. 73 cm

13. 180 cm^2

14. $(0,0,0), (1,0,-1), (0,-1,1), (-1,1,0)$

15. $(-\infty, -3) \cup (\frac{1}{6}, +\infty)$

16. $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, +\infty)$

17. 1000000

18. $(-2, -2008)$

19. $f(x) = 1$ for all $x \in (0, +\infty)$

20. $(2,1), (-2,-1), (2, \frac{3}{7}), (-2, -\frac{3}{7})$

Problem 21. There is an assignment of the integers $1, 2, 3, \ldots, 9$ to the vertices of the regular nonagon that gives $n = 16$. Let S be the sum of all sums of the integers assigned to three consecutive vertices. If there are integers assigned to three consecutive vertices whose sum is at most 14, then $S < 135$, which is a contradiction. Thus, every sum of the integers assigned to three consecutive vertices is equal to 15. Consider a, b, c, d. Then $a + b + c = 15$ and $b + c + d = 15$, which implies that $a = d$, a contradiction again. □

Problem 22. Let EF intersect AG and AH at points I and J, respectively. Note that $EGHJ$ and $FIGH$ are parallelograms. It follows that $\triangle FHJ \cong \triangle IGE$, which implies that $FJ = EI$. Using three pairs of similar triangles, $FJ = EI$ implies that $BG = DH$.

Let S be the midpoint of AC. Since $AGCH$ is parallelogram, S is the midpoint of GH. Finally, with $BG = DH$, we have $BS = BG + GS = DH + HS = DS$. This means that the diagonals AC and BD bisect each other, and so $ABCD$ is a parallelogram. □

Problem 23. Let r and s be two different integral roots of $x^3 + ax^2 + bx + c = 0$; that is, $r^3 + ar^2 + br + c = 0$ and $s^3 + as^2 + bs + c = 0$. Since p divides a, b, and c, it follows that p divides both r^3 and s^3. Being prime, p divides r and s.

Subtracting the above equations involving r and s, we get

$$r^3 - s^3 + a(r^2 - s^2) + b(r - s) = 0, \quad \text{or}(r - s)(r^2 + rs + s^2 + a(r + s) + b) = 0.$$

Since $r \neq s$, the last equation becomes

$$r^2 + rs + s^2 + a(r + s) + b = 0.$$

Because the terms (other than b) are divisible by p^2, the last equation forces p^2 to divide b.

Finally, the terms (other than c) of $r^3 + ar^2 + br + c = 0$ are divisible by p^3, it follows that p^3 divides c. □