18^{th} Philippine Mathematical Olympiad

Area Stage

14 November 2015

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. Marc and Jon together have 66 marbles although Marc has twice as many marbles as Jon. Incidentally, Jon found a bag of marbles which enabled him to have three times as many marbles as Mark. How many marbles were in the bag that Jon found?

2. A camera's aperture determines the size of the circular opening in the lens that allows light in. If we want to allow twice as much light in, what should be the ratio of the new radius to the current radius?

3. Determine all values of $k \in \mathbb{R}$ for which the equation
$$\frac{4(2015^x) - 2015^{-x}}{2015^x - 3(2015^{-x})} = k$$
admits a real solution.

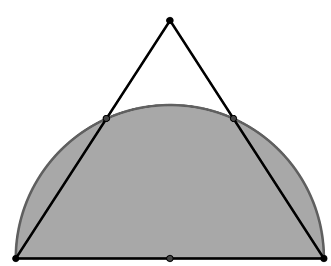
4. The points $(3, m)$, (x_1, y_1) and (x_2, y_2) are on the graph of the function $f(x) = \log_a x$. If $y_1 + y_2 = 2m$, find the value of $x_1 x_2$.

5. Let a, b, and c be three consecutive even numbers such that $a > b > c$. What is the value of $a^2 + b^2 + c^2 - ab - bc - ac$?

6. Evaluate
$$\prod_{\theta=1}^{89}(\tan\theta^\circ \cos 1^\circ + \sin 1^\circ).$$

7. Find the sum of all the prime factors of 27,000,001.

8. Refer to the figure on the right. A side of an equilateral triangle is the diameter of the given semi-circle. If the radius of the semi-circle is 1, find the area of the unshaded region inside the triangle.



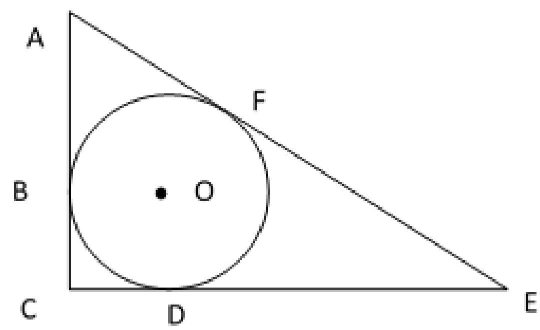
(Figure is not drawn to scale.)

9. How many ways can you place 10 identical balls in 3 baskets of different colors if it is possible for a basket to be empty?

10. Find the largest number N so that

$$\sum_{n=5}^{N} \frac{1}{n(n-2)} < \frac{1}{4}.$$

11. Refer to the figure below. If circle O is inscribed in the right triangle ACE as shown below, and if the length of AB is twice the length of BC, find the length of CE if the perimeter of the right triangle is 36 units.



12. Find all real solutions to the system of equations

$$\begin{cases} x(y-1)+y(x+1) & = & 6, \\ (x-1)(y+1) & = & 1. \end{cases}$$

13. Find all real numbers a and b so that for all real numbers x,

$$2\cos^2\left(x+\frac{b}{2}\right) - 2\sin\left(ax-\frac{\pi}{2}\right)\cos\left(ax-\frac{\pi}{2}\right) = 1.$$

14. Let P be the product of all prime numbers less than 90. Find the largest integer N so that for each $n \in \{2, 3, 4, ..., N\}$, the number $P+n$ has a prime factor less than 90.

15. In how many ways can the letters of the word ALGEBRA be arranged if the order of the vowels must remain unchanged?

16. The lengths of the sides of a rectangle are all integers. Four times its perimeter is numerically equal to one less than its area. Find the largest possible perimeter of such a rectangle.

17. Find the area of the region bounded by the graph of $|x| + |y| = \frac{1}{4}|x+15|$.

18. Given $f(1-x) + (1-x)f(x) = 5$ for all real number x, find the maximum value that is attained by $f(x)$.

19. The amount 4.5 is split into two nonnegative real numbers uniformly at random. Then each number is rounded to its nearest integer. For instance, if 4.5 is split into $\sqrt{2}$ and $4.5-\sqrt{2}$, then the resulting integers are 1 and 3, respectively. What is the probability that the two integers sum up to 5?

20. Let s_n be the sum of the digits of a natural number n. Find the smallest value of $\dfrac{n}{s_n}$ if n is a four-digit number.

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. The 6-digit number $739ABC$ is divisible by 7, 8, and 9. What values can A, B, and C take?

2. The numbers from 1 to 36 can be written in a counterclockwise spiral as follows:

$$\begin{array}{cccccc}
\boxed{31} & 30 & 29 & 28 & 27 & 26 \\
32 & \boxed{13} & 12 & 11 & 10 & 25 \\
33 & 14 & \boxed{3} & 2 & 9 & 24 \\
34 & 15 & 4 & \boxed{1} & 8 & 23 \\
35 & 16 & 5 & 6 & \boxed{7} & 22 \\
36 & 17 & 18 & 19 & 20 & \boxed{21}
\end{array}$$

In the figure above, all the terms on the diagonal beginning from the upper left corner have been enclosed in a box, and these entries sum up to 76.

Suppose this spiral is continued all the way until 2015, leaving an incomplete square. Find the sum of all the terms on the diagonal beginning from the upper left corner of the resulting (incomplete) square.

3. Point P on side BC of triangle ABC satisfies

$$|BP| : |PC| = 2 : 1.$$

Prove that the line AP bisects the median of triangle ABC drawn from vertex C.

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PART I. (3 points each)

1. 110 marbles
2. $\sqrt{2}$
3. $k \in (-\infty, \frac{1}{3}) \cup (4, +\infty)$
4. 9
5. 12
6. $\csc 1^\circ$ or $\sec 89^\circ$ or equivalent
7. 652
8. $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$
9. 66
10. 24
11. 12
12. $(4/3, 2), (-2, -4/3)$
13. $a = 1$ and $b = -\frac{3\pi}{2} + 2k\pi$, or $a = -1$ and $b = \frac{3\pi}{2} + 2k\pi$ where $k \in \mathbb{Z}$
14. 96
15. 840
16. 164 units
17. 30
18. 5
19. 4/9
20. $\dfrac{1099}{19}$

PART II. (10 points each)

1. Since $\gcd(7, 8, 9) = 1$, then $739ABC$ is divisible by $7, 8$, and 9 iff it is divisible by $7 \cdot 8 \cdot 9 = 504$. Note that the only integers between 739000 and 739999 which are divisible by 504 are 739368 and 739872. So, $(A, B, C) \in \{(3, 6, 8), (8, 7, 2)\}$.

2. Solution 1:

   The closest perfect square to 2015 is $2025 = 45^2$ which means that only the rightmost side will be incomplete while the required diagonal would still have a total of 45 entries.

   Looking at the values on the diagonal, we see that the numbers on the diagonal above 1 have a common second dierence. This suggests that this sequence satisfies a quadratic function of the form $f(n) = an^2 + bn + c$. Since $f(1) = 1$, $f(2) = 3$, $f(3) = 13$, solving a simple system of three equations gives us $f(n) = 4n^2 - 10n + 7$, $1 \le n \le 23$. On the other hand, the numbers on the diagonal below 1 also have a common second difference. This gives a sequence $g(n) = dn^2 + en + f$ with $g(1) = 1$, $g(2) = 7$, and $g(3) = 21$. By solving a similar system as above, we obtain $g(n) = 4n^2 - 6n + 3$, where $1 \le n \le 23$. Taking the sum of these two sequences of numbers, we have

$$
\begin{aligned}
\sum_{n=1}^{23}[f(n)+g(n)] &= \sum_{n=1}^{23}\left(8n^2-16n+10\right)\\
&= 8\sum_{n=1}^{23}n^2-16\sum_{n=1}^{23}n+\sum_{n=1}^{23}10\\
&= 8\left[\frac{(23)(24)(47)}{6}\right]-168\left[\frac{(23)(24)}{2}\right]+10(23)\\
&= 30,406
\end{aligned}
$$

Since 1 is counted twice, the required sum must be $30,406-1=30,405$.

Solution 2:

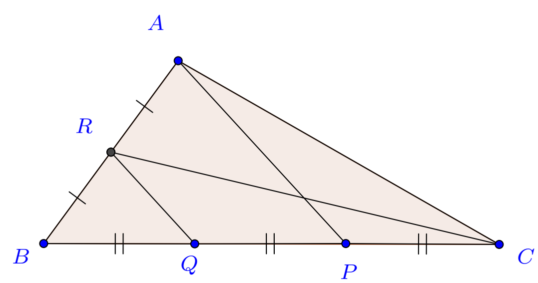
Filling the square with a few more numbers enables us to see that the boxed numbers

$$1,3,7,13,21,31,\ldots,1981$$

satisfy the recurrence relation $a_1=1$ and $(\forall n\in\mathbb{N})\ a_{n+1}=a_n+2n$. The associated homogeneous recurrence relation is solved by $a_n^{(h)}\equiv 1$. Testing a particular solution of the form $a_n^{(p)}=n(cn+d)$, we see that $c=1$ and $d=-1$. Therefore, the solution to the nonhomogeneous recurrence relation is $a_n=n^2-n+1$. The last boxed number 1981 corresponds to $n=45$. Therefore,

$$\sum_{n=1}^{45}(n^2-n+1)=\frac{45\cdot 46\cdot 91}{6}-\frac{45\cdot 46}{2}+45=30,405.$$

3. Solution 1: Let Q be the midpoint of line segment BP. The conditions of the problem imply $|BQ|=|QP|=|PC|=\frac{1}{3}|BC|$. Let R be the midpoint of line segment AB. Then RQ is a midline of ABP. Consequently, $RQ\,||\,AP$. Ray AP bisects side CQ of triangle CRQ while being parallel to side RQ of this triangle. Thus AP extends the midline of triangle CRQ and bisects therefore also its side CR. But line segment CR is the median of triangle ABC drawn from vertex C.



Solution 2: Let R be the midpoint of segment AB. Choose point D on ray AC beyond point C such that $|AC|=|CD|$. Then BC is a median of triangle ABD. As $|BP|:|PC|=2:1$, point P is the intersection point of medians of triangle ABD. Thus AP lies entirely on the other median of triangle ABD, i.e., ray AP bisects the segment BD. As CR is the midline of triangle ABD, we have $CR\,||\,BD$, implying that ray AP also bisects the segment CR. But this is the median of triangle ABC drawn from vertex C.