17^{th} Philippine Mathematical Olympiad

Area Stage

15 November 2014

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. What is the fourth smallest positive integer having exactly 4 positive integer divisors, including 1 and itself?

2. Let $f(x) = a^x - 1$. Find the largest value of $a > 1$ so that if $0 \leq x \leq 3$, then $0 \leq f(x) \leq 3$.

3. Simplify the expression $\left(1 + \frac{1}{i} + \frac{1}{i^2} + \ldots + \frac{1}{i^{2014}}\right)^2$.

4. Find the numerical value of $(1 - \cot 37°)(1 - \cot 8°)$.

5. Triangle ABC has a right angle at B, with $AB = 3$ and $BC = 4$. If D and $E$are points on AC and BC, respectively, such that $CD = DE = \frac{5}{3}$, find the perimeter of quadrilateral $ABED$.

6. Rationalize the denominator of $\dfrac{6}{\sqrt[3]{4} + \sqrt[3]{16} + \sqrt[3]{64}}$ and simplify.

7. Find the area of the triangle having vertices $A(10, -9)$, $B(19, 3)$, and $C(25, -21)$.

8. How many ways can 6 boys and 6 girls be seated in a circle so that no two boys sit next to each other?

9. Two numbers p and q are both chosen randomly (and independently of each other) from the interval $[-2, 2]$. Find the probability that $4x^2 + 4px + 1 - q^2 = 0$ has imaginary roots.

10. In $\triangle ABC$, $\angle A = 80°$, $\angle B = 30°$, and $\angle C = 70°$. Let BH be an altitude of the triangle. Extend BH to a point D on the other side of AC so that $BD = BC$. Find $\angle BDA$.

11. Find all integer values of n that will make $\dfrac{6n^3 - n^2 + 2n + 32}{3n + 1}$ an integer.

12. Suppose that the function $y = f(x)$ satisfies $1 - y = \dfrac{9e^x + 2}{12e^x + 3}$. If m and n are consecutive integers so that $m < \dfrac{1}{y} < n$ for all real x, find the value of mn.

13. The product of the two roots of $\sqrt{2014}x^{\log_{2014} x} = x^{2014}$ is an integer. Find its units digit.

14. In how many ways can Alex, Billy, and Charles split 7 identical marbles among themselves so that no two have the same number of marbles? It is possible for someone not to get any marbles.

15. In a Word Finding game, a player tries to find a word in a 12×12 array of letters by looking at blocks of adjacent letters that are arranged horizontally, arranged vertically, or arranged diagonally. How many such 3-letter blocks are there in a given 12×12 array of letters?

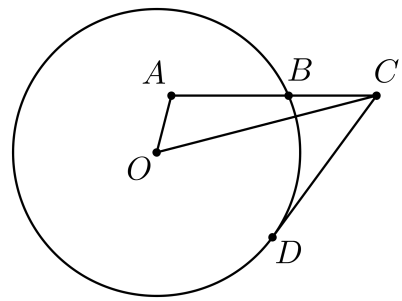
16. Find the largest possible value of

$$(\sin\theta_1)(\cos\theta_2) + (\sin\theta_2)(\cos\theta_3) + \cdots + (\sin\theta_{2013})(\cos\theta_{2014}) + (\sin\theta_{2014})(\cos\theta_1).$$

17. What is the remainder when

$$16^{15} - 8^{15} - 4^{15} - 2^{15} - 1^{15}$$

is divided by 96?



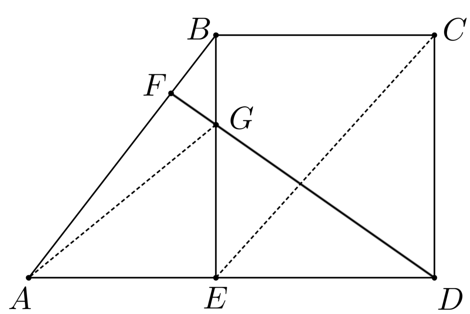
18. Segment CD is tangent to the circle with center O, at D. Point A is in the interior of the circle, and segment AC intersects the circle at B. If $OA = 2$, $AB = 4$, $BC = 3$, and $CD = 6$, find the length of segment OC.

19. Find the maximum value of

$$(1-x)(2-y)(3-z)\left(x + \frac{y}{2} + \frac{z}{3}\right)$$

where $x < 1$, $y < 2$, $z < 3$, and $x + \frac{y}{2} + \frac{z}{3} > 0$.

20. Trapezoid $ABCD$ has right angles at C and D, and $AD > BC$. Let E and F be the points on AD and AB, respectively, such that $\angle BED$ and $\angle DFA$ are right angles. Let G be the point of intersection of segments BE and DF. If $\angle CED = 58°$ and $\angle FDE = 41°$, what is $\angle GAB$?



PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. Arrange these four numbers from smallest to largest: $\log_3 2$, $\log_5 3$, $\log_{625} 75$, $\frac{2}{3}$.

2. What is the greatest common factor of all integers of the form $p^4 - 1$, where p is a prime number greater than 5?

3. Points A, M, N and B are collinear, in that order, and $AM = 4$, $MN = 2$, $NB = 3$. If point C is not collinear with these four points, and $AC = 6$, prove that CN bisects $\angle BCM$.

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1. 14
2. $\sqrt[3]{4}$
3. -1
4. 2
5. $\dfrac{28}{3} = 9\dfrac{1}{3}$
6. $2 - \sqrt[3]{2}$
7. 144
8. $6!5! = 86400$
9. $\dfrac{\pi}{16}$
10. 70°
11. 0, 10
12. 12
13. 6
14. 24
15. 440
16. 1007
17. 31
18. $\sqrt{60} = 2\sqrt{15}$
19. $\dfrac{3^5}{2^7} = \dfrac{243}{128}$
20. 17°

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. Arrange these four numbers from smallest to largest: $\log_3 2$, $\log_5 3$, $\log_{625} 75$, $\frac{2}{3}$.
   Solution: The numbers, arranged from smallest to largest, are $\log_3 2$, $\frac{2}{3}$, $\log_{625} 75$, and $\log_5 3$.

   - Since $\left(3^{\log_3 2}\right)^3 = 8$ and $\left(3^{\frac{2}{3}}\right)^3 = 9$, then $\log_3 2 < \frac{2}{3}$.
   - Since $\left(625^{\frac{2}{3}}\right)^3 = 5^8 = 5^6 \cdot 25$ and $\left(625^{\log_{625} 75}\right)^3 = 75^3 = 5^6 \cdot 27$, then $\frac{2}{3} < \log_{625} 75$.
   - If $A = \log_{625} 75$, then $5^{4A} = 75$. On the other hand, $5^{4\log_5 3} = 81$. Thus, $\log_{625} 75 < \log_5 3$.

2. What is the greatest common factor of all integers of the form $p^4 - 1$, where p is a prime number greater than 5?
   Solution: Let $f(p) = p^4 - 1 = (p-1)(p+1)(p^2+1)$. Note that $f(7) = 2^5 \cdot 3 \cdot 5^2$ and $f(11) = 2^4 \cdot 3 \cdot 5 \cdot 61$. We now show that their greatest common factor, $2^4 \cdot 3 \cdot 5$, is actually the greatest common factor of all numbers $p^4 - 1$ so described.

   - Since p is odd, then $p^2 + 1$ is even. Both $p - 1$ and $p + 1$ are even, and since they are consecutive even integers, one is actually divisible by 4. Thus, $f(p)$ is always divisible by 2^4.
   - When divided by 3, p has remainder either 1 or 2.
     - If $p \equiv 1$, then $3 | p - 1$.
     - If $p \equiv 2$, then $3 | p + 1$.

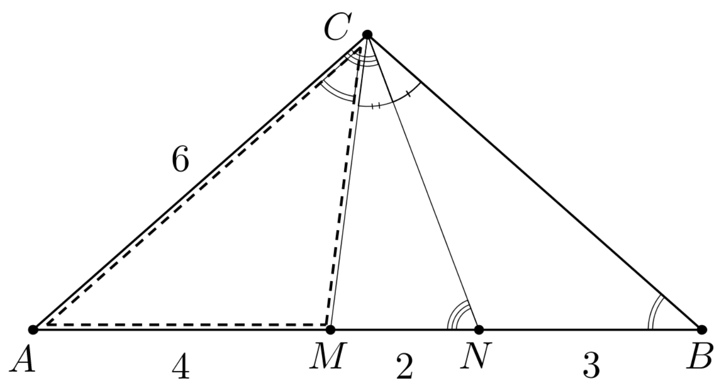
     Thus, $f(p)$ is always divisible by 3.

- When divided by 5, p has remainder 1, 2, 3 or 4.
  - If $p \equiv 1$, then $5|p-1$.
  - If $p \equiv 2$, then $p^2+1 \equiv 2^2+1 = 5 \equiv 0$.
  - If $p \equiv 3$, then $p^2+1 \equiv 3^2+1 = 10 \equiv 0$.
  - If $p \equiv 4$, then $5|p+1$.

  Thus, $f(p)$ is always divisible by 5.

Therefore, the greatest common factor is $2^4 \cdot 3 \cdot 5 = 240$.

3. Points A, M, N and B are collinear, in that order, and $AM = 4$, $MN = 2$, $NB = 3$. If point C is not collinear with these four points, and $AC = 6$, prove that CN bisects $\angle BCM$.
Solution:



Since $\dfrac{CA}{AM} = \dfrac{3}{2} = \dfrac{BA}{AC}$ and $\angle CAM = \angle BAC$, then $\triangle CAM \sim \triangle BAC$. Therefore,

$$\angle MCA = \angle CBA. \tag{1}$$

Since $AC = 6 = AN$, then $\triangle CAN$ is isosceles. Therefore,

$$\angle ACN = \angle ANC. \tag{2}$$

Thus,

$$\begin{aligned} \angle BCN &= \angle ANC - \angle CBA && \text{since } \angle ANC \text{ is an exterior angle of } \triangle BNC \\ &= \angle ACN - \angle MCA && \text{using (1) and (2)} \\ &= \angle MCN. \end{aligned}$$