PART I. Each correct answer is worth two points.

1. Find the sum of

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\cdots+\frac{1}{2009\times 2012}.$$

(a) $\frac{335}{2012}$ (b) $\frac{545}{2012}$ (c) $\frac{865}{2012}$ (d) $\frac{1005}{2012}$

2. Find the last two digits of $0!+5!+10!+15!+\cdots+100!$.

(a) 00 (b) 11 (c) 21 (d) 01

3. Consider the system

$$xy=10^a, yz=10^b, xz=10^c.$$

What is $\log x+\log y+\log z$?

(a) $\frac{abc}{2}$ (b) $\frac{a+b+c}{2}$ (c) $a+b+c$ (d) abc

4. A polyhedron has 30 faces and 62 edges. How many vertices does the polyhedron have?

(a) 61 (b) 34 (c) 46 (d) 77

5. Which of the following quadratic expressions in x have roots $\frac{g}{h}$ and $-\frac{h}{g}$?

(a) $g^2h^2x^2-\frac{g^2}{h^2}$
(b) $hgx^2+(g^2-h^2)x-hg$
(c) $hgx^2+(h^2-g^2)x+hg$
(d) $hgx^2+(h^2-g^2)x-hg$

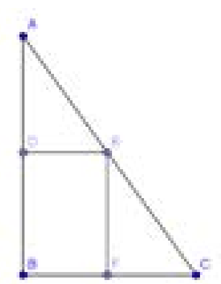
6. If $x^6=64$ and $\left(\frac{2}{x}-\frac{x}{2}\right)^2=b$, then a function f that satisfies $f(b+1)=0$ is

(a) $f(x)=1-2^{x-1}$
(b) $f(x)=2^{x-1}$
(c) $f(x)=x^2+x$
(d) $f(x)=2x-1$

7. Sixty men working on a construction job have done 1/3 of the work in 18 days. The project is behind schedule and must be accomplished in the next twelve days. How many more workers need to be hired?

   (a) 60 (b) 180 (c) 120 (d) 240

8. The vertices D, E and F of the rectangle are midpoints of the sides of $\triangle ABC$. If the area of $\triangle ABC$ is 48, find the area of the rectangle.



   (a) 12 (b) 24 (c) 6 (d) $12\sqrt{2}$

9. Determine the number of factors of $5^x + 2 \cdot 5^{x+1}$.

   (a) x (b) $x+1$ (c) $2x$ (d) $2x+2$

10. How many solutions has $\sin 2\theta - \cos 2\theta = \sqrt{6}/2$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

   (a) 1 (b) 2 (c) 3 (d) 4

11. If $2\sin(3x) = a\cos(3x+c)$, find all values of ac. In the choices below, k runs through all integers.

   (a) $-\frac{\pi}{2}$ (c) $-\pi$
   (b) $2k\pi$ (d) $(4k-1)\pi$

12. If x satisfies $\dfrac{\log_2 x}{\log_2 2x - \log_8 2} = 3$, the value of $1 + x + x^2 + x^3 + x^4 + \cdots$ is

   (a) 1 (c) $\frac{1}{2}$
   (b) 2 (d) the value does not exist

13. Find the least common multiple of 15! and $2^3 3^9 5^4 7^1$.

(a) $2^3 3^6 5^3 7^1 11^1 13^1$
(b) $2^3 3^6 5^3 7^1$
(c) $2^{11} 3^9 5^4 7^2 11^1 13^1$
(d) $2^{11} 3^9 5^4 7^2$

14. If (a, b) is the solution of the system $\sqrt{x+y} + \sqrt{x-y} = 4$, $x^2 - y^2 = 9$, then $\dfrac{ab}{a+b}$ has value

(a) $\dfrac{10}{9}$ (b) $\dfrac{8}{3}$ (c) 10 (d) $\dfrac{20}{9}$

15. Find the value of $\sin\theta$ if the terminal side of θ lies on the line $5y - 3x = 0$ and θ is in the first quadrant.

(a) $\dfrac{3}{\sqrt{34}}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{3}{5}$ (d) $\dfrac{4}{\sqrt{34}}$

PART II. Each correct answer is worth three points.

1. Find the value of $\log_2[2^3 4^4 8^5 \cdots (2^{20})^{22}]$.

(a) 3290 (b) 3500 (c) 3710 (d) 4172

2. Let $2 = \log_b x$. Find all values of $\dfrac{x+1}{x}$ as b ranges over all positive real numbers.

(a) $(0, +\infty)$
(b) $(1, +\infty)$
(c) $(0, 1)$
(d) all real numbers

3. Solve the inequality $\dfrac{1}{3^x}\left(\dfrac{1}{3^x} - 2\right) < 15$.

(a) $\left(-\dfrac{\log 5}{\log 3}, +\infty\right)$
(b) $\left(-\infty, \dfrac{\log 5}{\log 3}\right)$
(c) $\left(\dfrac{\log 3}{\log 5}, 1\right)$
(d) $(\log 3, \log 5)$

4. Find the solution set of the inequality $\left(\frac{\pi}{2}\right)^{(x-1)^2} \leq \left(\frac{2}{\pi}\right)^{x^2-5x-5}$.

(a) $[-1, 4]$
(b) $(-\infty, -1] \cup (4, +\infty)$
(c) $[-1/2, 4]$
(d) $(-\infty, -1/2] \cup [4, +\infty)$

5. The equation $x^2 - bx + c = 0$ has two roots p and q. If the product pq is to be maximum, what value of b will make $b + c$ minimum?

(a) $-c$
(b) -2
(c) $\frac{1}{2}$
(d) $2c$

6. Find the range of the function $f(x) = \dfrac{6}{5\sqrt{x^2 - 10x + 29} - 2}$.

(a) $[-1/2, 3/4]$
(b) $(0, 3/4]$
(c) $(1/2, 4/3]$
(d) $[-1/2, 0) \cup (0, 4/3]$

7. A fair die is thrown three times. What is the probability that the largest outcome of the three throws is a 3?

(a) 1/36
(b) 19/216
(c) 25/36
(d) 1/8

8. If f is a real-valued function, defined for all nonzero real numbers, such that $f(a) + \dfrac{1}{f(b)} = f\left(\dfrac{1}{a}\right) + f(b)$, find all possible values of $f(1) - f(-1)$.

(a) $\{-2, 2\}$
(b) $\{0, -1, 1\}$
(c) $\{1, 2\}$
(d) $\{0, -2, 2\}$

9. Which is a set of factors of $(r - s)^3 + (s - t)^3 + (t - r)^3$?

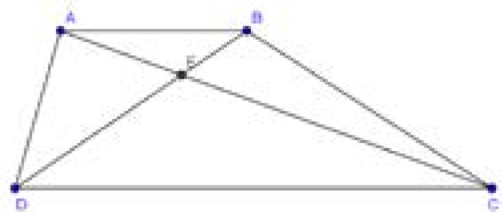
(a) $\{r - s, s - t, t - r\}$
(b) $\{3r - 3s, s + t, t + r\}$
(c) $\{r - s, s - t, t - 2rt + r\}$
(d) $\{r^2 - s^2, s - t, t - r\}$

10. If $36 - 4\sqrt{2} - 6\sqrt{3} + 12\sqrt{6} = (a\sqrt{2} + b\sqrt{3} + c)^2$, find the value of $a^2 + b^2 + c^2$.

(a) 12
(b) 5
(c) 14
(d) 6

PART III. Each correct answer is worth six points.

1. $ABCD$ is a trapezoid with $AB||CD$, $AB = 6$ and $CD = 15$. If the area of $\triangle AED = 30$, what is the area of $\triangle AEB$?



(a) 20 (b) 40/7 (c) 12 (d) 8/3

2. Find the maximum value of $y = (7-x)^4(2+x)^5$ when x lies strictly between -2 and 7.

(a) $7^4 2^5$
(b) $(4.5)^4(2.5)^5$
(c) $(2.5)^9$
(d) $(4.5)^9$

3. Find all possible values of $\dfrac{2 \cdot 3^{-x} - 1}{3^{-x} - 2}$, as x runs through all real numbers.

(a) $(-\infty, 1/2) \cup (2, +\infty)$
(b) $(1/2, 2)$
(c) $[2, +\infty]$
(d) $(0, +\infty)$

4. In how many ways can one select five books from a row of twelve books so that no two adjacent books are chosen?

(a) 34 (b) 78 (c) 42 (d) 56

5. Find the range of

$$f(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)}$$

where a, b, c are distinct real numbers.

(a) all real numbers
(b) $\{1\}$
(c) $[-a-b-c, +\infty)$
(d) $\{a+b+c\}$

Part I. No solution is needed. All answers must be in simplest form. Each correct answer is worth three points.

1. Find all complex numbers z such that $\dfrac{z^4+1}{z^4-1} = \dfrac{i}{\sqrt{3}}$.

2. Find the remainder if $(2001)^{2012}$ is divided by 10^6.

3. If $z^3 - 1 = 0$ and $z \neq 1$, find the value of $z + \dfrac{1}{z} + 4$.

4. Find the equation of the line that contains the point $(1, 0)$, that is of least positive slope, and that does not intersect the curve $4x^2 - y^2 - 8x = 12$.

5. Consider a function $f(x) = ax^2+bx+c, a > 0$ with two distinct roots a distance p apart. By how much, in terms of a, b, c should the function be translated downwards so that the distance between the roots becomes $2p$?

6. Find the equation of a circle, in the form $(x - h)^2 + (y - k)^2 = r^2$, inscribed in a triangle whose vertex are located at the points $(-2, 1), (2, 5), (5, 2)$.

7. Define $f(x) = \dfrac{a^x}{a^x + \sqrt{a}}$ for any $a > 0$. Evaluate $\displaystyle\sum_{i=1}^{2012} f\left(\frac{i}{2013}\right)$.

8. Let $3x, 4y, 5z$ form a geometric sequence while $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ form an arithmetic sequence. Find the value of $\dfrac{x}{z} + \dfrac{z}{x}$.

9. Consider an acute triangle with angles α, β, γ opposite the sides a, b, c, respectively. If $\sin\alpha = \dfrac{3}{5}$ and $\cos\beta = \dfrac{5}{13}$, evaluate $\dfrac{a^2 + b^2 - c^2}{ab}$.

10. A change from Cartesian to polar coordinates involves the following transformation: $x = r\cos\theta$ and $y = r\sin\theta$. For a circle with polar equation $r = \dbinom{m}{n}\cos\theta$, where $1 \le n \le m \le 6$, how many distinct combinations of m and n will this equation represent a circle of radius greater than or equal to 5?

11. Let f be a polynomial function that satisfies $f(x - 5) = -3x^2 + 45x - 108$. Find the roots of $f(x)$.

12. Six boy-girl pairs are to be formed from a group of six boys and six girls. In how many ways can this be done?

13. From the xy-plane, select five distinct points that have integer coordinates. Find the probability that there is a pair of points among the five whose midpoint has integer coordinates.

14. Given that $\tan\alpha + \cot\alpha = 4$, find $\sqrt{\sec^2\alpha + \csc^2\alpha - \frac{1}{2}\sec\alpha\csc\alpha}$.

15. There are 100 people in a room. 60 of them claim to be good at math, but only 50 are actually good at math. If 30 of them correctly deny that they are good at math, how many people are good at math but refuse to admit it?

16. Find the value of p, where,

$$p = \frac{16^2 - 4}{18 \times 13} \times \frac{16^2 - 9}{19 \times 12} \times \frac{16^2 - 16}{20 \times 11} \times \cdots \times \frac{16^2 - 64}{24 \times 7}.$$

17. The number x is chosen randomly from the interval $(0, 1]$. Define $y = \lceil \log_4 x \rceil$. Find the sum of the lengths of all subintervals of $(0, 1]$ for which y is odd. For any real number a, $\lceil a \rceil$ is defined as the smallest integer not less than a.

18. Find the value/s of k so that the inequality $k(x^2 + 6x - k)(x^2 + x - 12) > 0$ has solution set $(-4, 3)$.

19. A sequence of numbers is defined using the relation

$$a_n = -a_{n-1} + 6a_{n-2}$$

where $a_1 = 2, a_2 = 1$. Find $a_{100} + 3a_{99}$.

20. Define the following operation for real numbers: $a \star b = ab + a + b$. If $x \star y = 11$, $y \star z = -4$, and $x \star z = -5$. What is the difference between the maximum and minimum elements of the solution set $\{x, y, z\}$?

Part II. Show the solution to each item. Each complete and correct solution is worth ten points.

1. If $x + y + xy = 1$, where x, y are nonzero real numbers, find the value of

$$xy + \frac{1}{xy} - \frac{y}{x} - \frac{x}{y}.$$

2. The quartic (4th-degree) polynomial $P(x)$ satisfies $P(1) = 0$ and attains its maximum value of 3 at both $x = 2$ and $x = 3$. Compute $P(5)$.

3. Let $v(X)$ be the sum of elements of a nonempty finite set X, where X is a set of numbers. Calculate the sum of all numbers $v(X)$ where X ranges over all nonempty subsets of the set $\{1, 2, 3, ..., 16\}$.

ANSWER KEY

Qualifying Round

I.
1. A
2. C
3. B
4. B
5. D
6. A
7. C
8. B
9. D
10. D
11. D
12. B
13. C
14. D
15. A

II.
1. A
2. B
3. A
4. C
5. B
6. B
7. B
8. D
9. A
10. C

III.
1. C
2. D
3. A
4. D
5. B

Area Stage

I.
1. $\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{\sqrt{3}}{2}+\frac{1}{2}i,$ $-\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \frac{\sqrt{3}}{2}-\frac{1}{2}i$
2. 24001
3. 3
4. $y = 2x - 2$
5. $\dfrac{3b^2}{4a} - 3c$
6. $(x-2)^2 + (y-3)^2 = 2$
7. 1006
8. $\dfrac{34}{15}$
9. $\dfrac{32}{65}$
10. 5
11. 7, -2
12. 6!
13. 1
14. $\sqrt{14}$
15. 10
16. 2
17. 1/5
18. $(-\infty, -9]$
19. 7×2^{98}
20. 5

II.
1. Observe that

$$\begin{aligned} xy + \frac{1}{xy} - \frac{y}{x} - \frac{x}{y} &= \frac{(xy)^2 + 1 - x^2 - y^2}{xy} \\ &= \frac{(x^2-1)(y^2-1)}{xy} \\ &= \frac{(x+1)(y+1)(x-1)(y-1)}{xy} \\ &= (xy+x+y+1)\frac{(xy-x-y+1)}{xy}. \end{aligned}$$

Since, $xy+x+y=1$, the first term will equal to 2. Moreover, dividing both sides of the equation $xy+x+y=1$ by xy, we obtain

$$1+\frac{1}{y}+\frac{1}{x}=\frac{1}{xy},$$

which is equivalent to

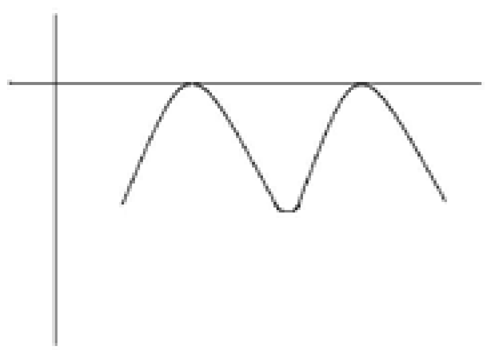
$$1=\frac{1}{xy}-\frac{1}{y}-\frac{1}{x}.$$

Hence, $(xy+x+y+1)\dfrac{(xy-x-y+1)}{xy}=2\cdot 2=\boxed{4}$.

2. Consider the polynomial $Q(x)=P(x)-3$. Then $Q(x)$ has zeros at and maximum value 0 at $x=2,3$. These conditions imply that $Q(x)$ has the form

$$Q(x)=A(x-2)^2(x-3)^2.$$

That is, its graph looks like



because the values of $Q(x)$ should grow larger and larger through negative values as the variable x goes to larger and larger values of both signs and the fact that the number of turning points should not exceed $4-1=3$ but should be more than 2 (given by the maximum points). Thus, $Q(1)=-3$ imply $a=-\dfrac{3}{4}$. Finally, $Q(5)=P(5)-3=-27$ and $P(5)=\boxed{-24}$.

3. The answer is $2^{15}\cdot 8\cdot 17$

We note that each $k\in\{1,2,3,...,16\}$ belongs to 2^{15} subsets of $\{1,2,3,...,16\}$. We reason as follows: we can assign 0 or 1 to k according to whether it is not or in a subset of {1, 2, 3,..., 16}. As there are 2 choices for a fixed k, k belongs to half of the total number of subsets, which is 2^{16}. Hence the sum is

$$\sum v(X)=2^{15}(1+2+3+\cdots+16)=2^{15}\cdot 8\cdot 17=\boxed{4456448}$$

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