15^{th} Philippine Mathematical Olympiad

National Stage, Oral Phase

26 January 2013

PART I. Each correct answer is worth two points.

1. In solving a problem that leads to a quadratic equation, one student made a mistake in the constant term only, obtaining the roots 8 and 2, while another student made a mistake in the coefficient of the first degree term, obtaining the roots -9 and -1. What was the original equation?

2. Determine the total number of integer pair solutions (x, y) that satisfy $x^2 + |y| = 2013$.

3. Find the values of a such that the system
$$x + 2y = a + 6,\ 2x - y = 25 - 2a$$
has a positive integer pair solution (x, y).

4. If $25 \cdot 9^{2x} = 15^y$ is an equality of integers, what is the value of x?

5. Find the solution set of the equation $(\sqrt{2} - 1)^x + 8(\sqrt{2} + 1)^x = 9$.

6. A function f defined on the set of real numbers has the property $f(a + b) = f(a)f(b)$. If $f(4) = 625$, what is the value of $3f(-2)$?

7. Let $P(x) = ax^7 + bx^3 + cx - 5$, where a, b and c are constants. If $P(-7) = 7$, find the value of $P(7)$.

8. If $7x + 4y = 5$ and x and y are integers, find the value of $[\![y/x]\!]$.

9. If $x^2 + 2x + 5$ is a factor of $x^4 + ax^2 + b$, find the sum $a + b$.

10. If $\dfrac{x}{y} + \dfrac{y}{x} = 4$ and $xy = 3$ find the value of $xy(x + y)^2 - 2x^2y^2$.

11. In ΔABC, $\angle A = 60°$, $\angle B = 45°$, and $AC = \sqrt{2}$. Find the area of the triangle.

12. What is the length of the shortest path that begins at the point $(-3, 7)$, touches the $x-$axis, and then ends at a point on the circle $(x - 5)^2 + (y - 8)^2 = 25$?

13. Find the solution set of the equation

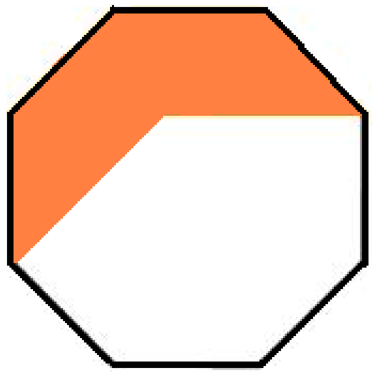
$$\frac{\sec^2 x - 6\tan x + 7}{\sec^2 x - 5} = 2$$

14. Find x if $\dfrac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}} = \dfrac{4x-1}{2}$

15. Who was the mathematician who discovered the existence of an "infinity of infinities", formalizing the idea through his definitions of cardinal and ordinal numbers and their arithmetic. He suffered numerous bouts of depression in his final years due to the rejection of his original ideas by important mathematicians of his time.

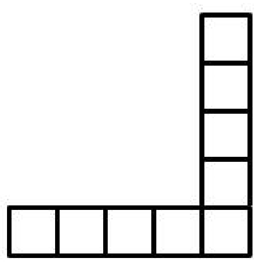
PART II. Each correct answer is worth three points.

1. How many pairs of diagonals on the surface of a rectangular prism are skew?
2. Let $f(g(x)) = \dfrac{3}{x+2}$ and $f(x) = \ln x + 2$, find the inverse $g^{-1}(x)$.
3. Find the area of the domain of the function $f(x,y) = \sqrt{25 - x^2 - y^2} - \sqrt{|x| - y}$.
4. Find the value of the infinite sum $1+1+3(1/2)^2+4(1/2)^3+5(1/2)^4+\cdots$
5. Suppose S is a subset of $\{1,2,3,4,5,6,7,8,9\}$. How many different possible values are there for the product of the elements in S?
6. If $\sqrt[3]{a+\sqrt{b}} = 12+\sqrt{5}$, find the value of $\sqrt[3]{a-\sqrt{b}}$.
7. Twenty-five people sit around a circular table. Three of them are chosen randomly. What is the probability that two of the three are sitting next to each other?
8. There are values of m for which $x^2 - 2x(1+3m) + 7(3+2m) = 0$ has equal roots. What are these equal roots?
9. Let $A(-3,0), B(3,0), C(0,5)$ and $D(0,-5)$. How many points $P(x,y)$ on the plane satisfy the properties $PA+PB = 10$ and $|PC-PD| = 6$?
10. Find the fraction of the area of the shaded region with respect to the regular octagon below.

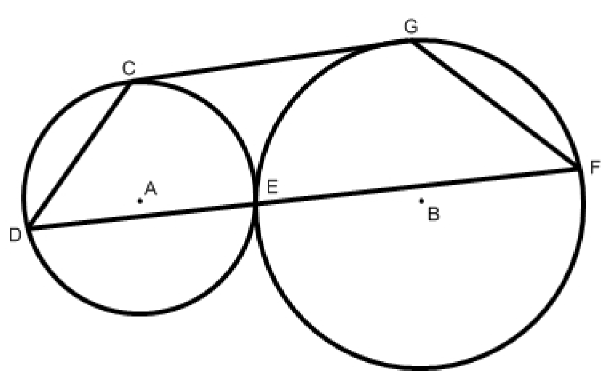


PART III. Each correct answer is worth six points.

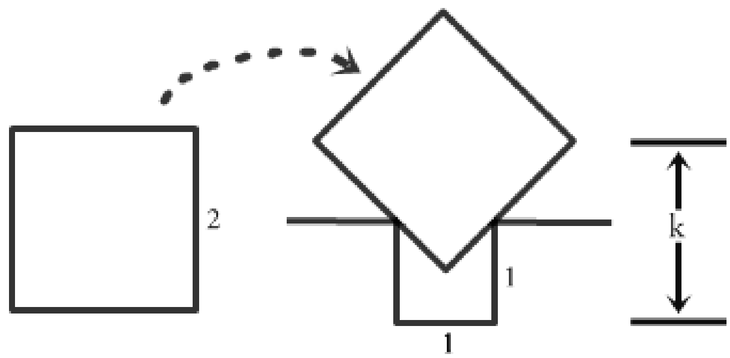
1. Each of the nine squares shown is to contain one number chosen from $1, 3, 5, 7, 9, 11, 13, 15,$ and 17 such that no number is used more than once. Suppose the sum of the squares aligned vertically is 53 and the sum of the squares aligned horizontally is 45. What number goes in the shared square?



2. Circles A and B are tangent to each other at E. The segment CG is tangent to both circles. A point D is selected on circle A and the segment DF is drawn such that it contains point E. If the measure of the minor arc CE is $102°$, find the measure of $\angle F$.



3. In the diagram below, the side of the square has length 2. Find the height k.



4. Find the least value of $a^6 + a^4 - a^3 - a + 1$.

5. Let $x = cy + bz$, $y = az + cx$, $z = bx + ay$. Find $\dfrac{(x-y)(y-z)(z-x)}{xyz}$ in terms of a, b, c.