PHILIPPINE MATHEMATICAL OLYMPIAD

MCMLXXXIV

PMO

PHILIPPINE MATHEMATICAL OLYMPIAD NATIONAL STAGE

Far Eastern University
18 January 2020

Registration	07:00 AM- 07:30 AM *Mini Auditorium*
Written Phase	07:30 AM - 12:00 NN *Room 910* *FEU Tech Building*
Lunch Break	12:00 NN - 01:00 PM
Oral Phase	01:00 PM - 05:00 PM *FEU Mini Auditorium*
Awarding Ceremonies & Dinner	06:30 PM - 09:00 PM *FEU Mini Auditorium*

ABOUT THE PMO

First held in **1984**, the PMO was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsman-ship. Its aims are: (1) to awaken greater interest in and promote the appreciation of mathematics among students and teachers; (2) to identify mathematically-gifted students and motivate them towards the development of their mathematical skills; (3) to provide a vehicle for the professional growth of teachers; and (4) to encourage the involvement of both public and private sectors in the promotion and development of mathematics education in the Philippines.

The PMO is the first part of the selection process leading to participation in the **International Mathematical Olympiad (IMO)**. It is followed by the **Mathematical Olympiad Summer Camp (MOSC)**, a five-phase program for the twenty two national finalists of PMO. The four selection tests given during the second phase of MOSC determine the tentative Philippine Team to the IMO. The final team is determined after the third phase of MOSC.

The PMO this year is the twenty-second since 1984. **Four thousand six hundred forty-nine (4649) junior and senior high school students** from about four hundred schools all over the country took the qualifying examination, out of these, **two hundred twenty-one (221) students** made it to the Area Stage. Now, in the National Stage, the number is down to **twenty-two (22)** and these twenty-two students will compete for the top three positions and hopefully move on to represent the country in the **61st International Mathematical Olympiad**, which will be held in **Saint Petersburg, Russia** from **08 to 18 July 2020**.

MESSAGE

Greetings of peace!

The past decade has been momentous for the Philippines in terms achieving milestones in science and mathematics education. Hence, there is no better way to welcome the new decade by hailing the best performers in the most prestigious mathematics competition in the country, the Philippine Mathematical Olympiad (PMO).

Now on its 22nd year, the PMO has persisted as the breeding ground for the most outstanding mathematics talents in the country and the world. Its goal of awakening greater interest and appreciation in the field of mathematics extends beyond gifted students and spreads to school officials, teachers, private entities, government institutions, and other young audiences, which helps our cause in building a culture of science across the nation. Truly, the PMO inspires us to keep pushing for more programs that will popularize science among the youth.

This year's national stage finals is expected to be tougher and for good reason—we are building on our international success in the past decade to do more in this new era. As the world has taken notice with our performances in the International Mathematical Olympiad (IMO), we expect to go hard in every iterations of the competition. We are confident that with the PMO, we continue our rise and that we continue to haul medals not as prizes but as collective tokens of success for the Filipino people.

The whole Department of Science and Technology – Science Education Institute (DOST-SEI) congratulates the Mathematical Society of the Philippines, our finalists, and all students and parents who made the 22nd PMO a success. We shall remain a consistent partner in promoting and developing mathematics education in the country.

We wish our participants all the best.

JOSETTE T. BIYO
Director
Science Education Institute
Department of Science and Technology

MESSAGE

Congratulations to all the 22 national stage finalists of the 22nd Philippine Mathematical Olympiad (PMO). Out of 4,649 students who joined the qualifying stage, you have made it this far. This year, we have 4,649 in the qualifying stage to 221 in the area stage and finally, to 22 in the national finals. Congratulations to the top scorers in Luzon, Visayas and Mindanao. Congratulations as well to the top scorers in their respective regions. Thanks to all the 4,649 students who took part in the 22nd PMO. I encourage you all to keep on training yourselves to be good mathematical problem solvers. I hope that you will all grow to become not only good mathematicians, engineers, or scientists for the country but also good persons with the strength and resilience to handle life's challenges. Thanks to all the coaches, their schools and the parents for guiding the students. I hope to see you all again in the 23rd PMO.

Successful mathematics problem solvers are developed over time. They must have that staying power to keep on trying to find a solution even when they have failed several times in their earlier attempts. They must also have the exposure to a wide spectrum of problems. Over the years, the PMO has given students exposure to many different types of problems that will sharpen their problem solving skills. Thanks to our excellent pool of problem writers (both past and present) who are also very good mathematicians in their own respective fields.

The PMO is an annual, national event that requires a good amount of time, expertise, money and other resources to work out all the nitty-gritty. Over the PMO's 22-year history, our past and present team of organizers have mastered all procedures that would make the conduct of every PMO smooth and seamless. Thanks to everyone that we have had in our team of organizers, especially the PMO Directors. This year, our PMO Director is Dr. Mary Anne Tirado of the Far Eastern University (FEU), which is the host of the 22nd PMO. Dr. Tirado works with her team composed of mathematicians from FEU, UP Diliman, De La Salle University, University of Santo Tomas and the Ateneo de Manila University. Many thanks as well to our regional and area coordinators.

The PMO would not be possible without the support from our sponsors. The Department of Science and Technology – Science Education's (DOST - SEI) and the Mathematical Society of the Philippines (MSP) have a continuing partnership to select the best among the best through the PMO and to conduct training for the country's participation in the International Mathematical Olympiad (IMO). Our heartfelt gratitude goes to Foundation for the Upgrading the Standard of Education Inc. (FUSE), SHARP and C&E whose generosity has certainly gone a long way in supporting the PMO. To Manulife Business Processing Services (MBPS) and Hyundai Asia Resources, Inc. (HARI) Foundation, we are thankful for the support for our IMO Team and the PMO.

On behalf of the MSP National Board of Directors, we would like to thank everyone who has contributed to the 22nd PMO a big success. Mabuhay po kayong lahat!

EMMANUEL A. CABRAL
President
Mathematical Society of the Philippines

MESSAGE

Congratulations to the 22nd Philippine Mathematical Olympiad!

Mathematics is a basic tool in life—indeed, we need to cultivate mathematical knowledge among the youth to nurture their skills in reasoning, abstract and critical thinking, problem-solving and even effective communication. The stages designed by PMO serve, in a way, to test how schools have developed these skills among our Filipino secondary school students. The types of questions formulated by the PMO Team indicate the value given nit just on accurate calculations but also on both the written and oral aspects related to the study of numbers, shapes, and patterns. The premium places on accuracy, different approaches, and hard work gives the participants the insight that there is always a solution to every problem, an insight that will downstream benefits to them in the real world.

Outside of the academic aspect, the cooperation engendered by the 22nd Mathematical Olympiad among public and private educational institutions and industry and business sectors gives the main message that partnerships can achieve objectives that are of value. I salute PMO for its accomplishments!

LUCIO C. TAN
Chairman
Foundation for Upgrading the Standard of Education, Inc.

MESSAGE

Mathematics truly has innovated our landscape of living and our team members from Sharp Calculators group are very much inspired to propel that vision alongside the Mathematical Society of the Philippines. Thus, we would like to extend our congratulations to this well-deserved success from Mathematical Society of the Philippines thru this 22nd Philippine Mathematical Olympiad Event.

It has always been a pleasure for our company, Sharp Calculators under Collins International Trading Corporation, to entrust our time and efforts in all your events and initiatives as our Filipino youth continuously bring pride and honor to our country. Congratulations to all the winners of this year's competition, also extending our gratitude to the people behind this amazing project. We assure our continuous support in many more fruitful years to come, most especially in all your programs that strives for quality in the field of Mathematics.

There are no secrets to success. It is the result of preparation, hard work, and learning from failure. — Colin Powell

God bless you!

SHERRY DONNE SABALVARO
National Sales and Marketing Manager
Collins International Trading Corporation
Sharp Calculators Philippines

MESSAGE

Chief Operating Officer
C&E Publishing, Inc.

THE PMO TEAM

DIRECTOR
May Anne Tirado

ASSISTANT DIRECTORS

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Ezra Aguilar

AREA STAGE
April Lynne Say-Awen

NATIONAL STAGE
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Richard Lemence
Josephine Tria Quiñones
Daryl Allen Saddi

REGIONAL COORDINATORS

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REGION II
Crizaldy Binarao

REGION III
Eduard Taganap

REGION IV-A
Sharon Lubag

REGION IV-B
Emmalyn Venturillo

REGION V
Glenda G. Quinto

REGION VI
Ferdinand Monoso

REGION VII
Lorna Almocera

REGION VIII
Oreste Ortega Jr.

REGION IX
Dante Partosa

REGION X, XII & ARMM
Emmy Chacon

REGION XI
Joseph Belida

REGION XIII
Miraluna Herrera

NCR
Mark Lexter de Lara
Lean Franzl Yao

AREA STAGE WINNERS

LUZON

1st	**BONA, Sarji Elijah**	Palawan Hope Christian School
2nd	**KHOO, Justin Teng Soon**	Regional Science High School III
3rd	**LIZARONDO, Marksen Victor**	Dasmariñas Integrated High School
3rd	**SARMIENTO, Justin**	Marcelo H. Del Pilar National High School

VISAYAS

1st	**KING, William Joshua**	University of San Carlos - North Campus (STEM)
1st	**YU, Jonathan Conrad**	University of San Carlos - North Campus (STEM)
3rd	**ANACAN, Jonathan**	Philippine Science High School - Western Visayas Campus

MINDANAO

1st	**CASIB, Mohammad Nur**	Philippine Science High School - Central Mindanao Campus
2nd	**TAN, Cassidy Kyler**	Davao Christian High School - V. Mapa Campus
2nd	**TY, Stephen James**	Zamboanga Chong Hua High School

NCR

1st	**DELA CRUZ, Vincent**	Valenzuela City School of Mathematics and Science
1st	**GONZALES, Andres Rico III**	De La Salle University Integrated School Manila
3rd	**ONG, Dion Stephan**	Ateneo de Manila Senior High School

PRIZES

The prizes for the **TOP THREE** for each **AREA** (Luzon, Visayas, Mindanao, NCR) are:

FIRST PLACE - Medal and SHARP Calculator
SECOND PLACE - Medal and SHARP Calculator
THIRD PLACE - Medal and SHARP Calculator

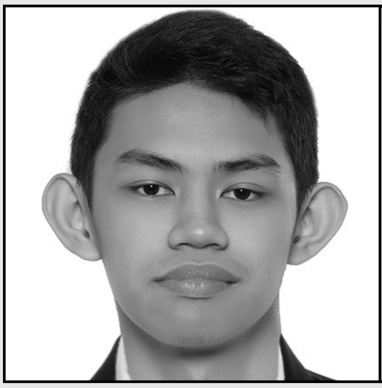
The prizes for the **TOP THREE** in the **NATIONAL STAGE** are:

CHAMPION - P 20,000, Medal, Trophy, and SHARP Calculator
FIRST RUNNER-UP - P 15,000, Medal, Trophy, and SHARP Calculator
SECOND RUNNER-UP - P 10,000, Medal, Trophy, and SHARP Calculator

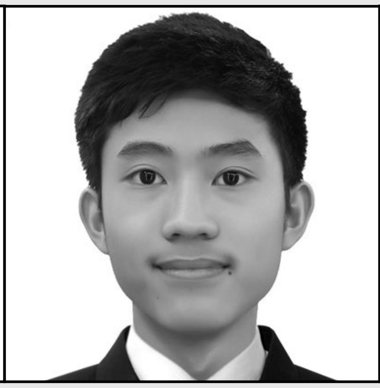
The coaches of the top three will receive P 5,000, P 3,000, and P 2,000, respectively.
The schools of the top three will receive trophies.

Each of the National Finalists will receive a medal and SHARP goodies.

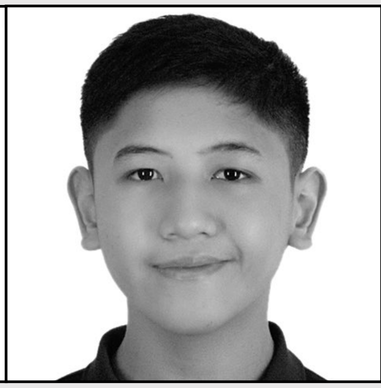
NATIONAL FINALISTS



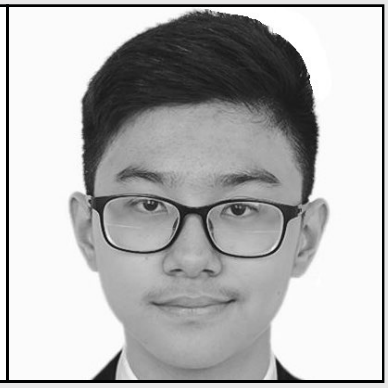
JOSE LORENZO
ABAD
Philippine Science High School - Main Campus



IMMANUEL JOSIAH
BALETE
Saint Stephen's High School

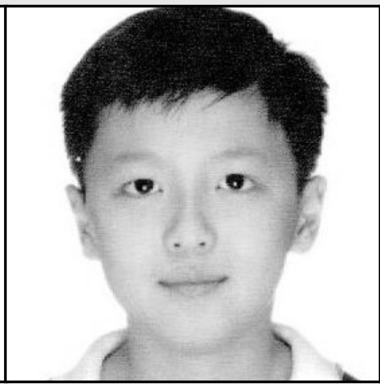


JOSE MARIA
BERNARDO II
Ateneo de Manila Junior High School



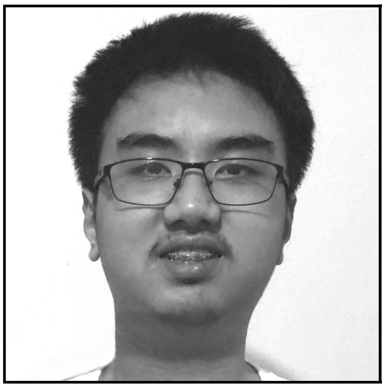
SARJI ELIJAH
BONA
Palawan Hope Christian School

EION NIKOLAI
CHUA
International School Manila

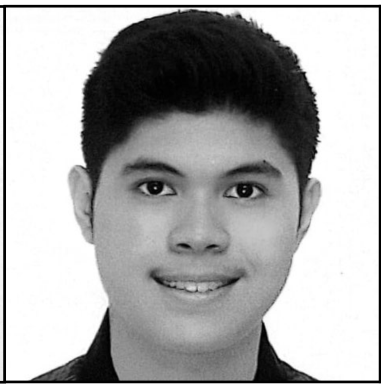


SHAWN DARREN
CHUA
MGC New Life Christian Academy

RAPHAEL DYLAN
DALIDA
Philippine Science High School - Main Campus



VINCENT
DELA CRUZ
Valenzuela City School of Mathematics and Science

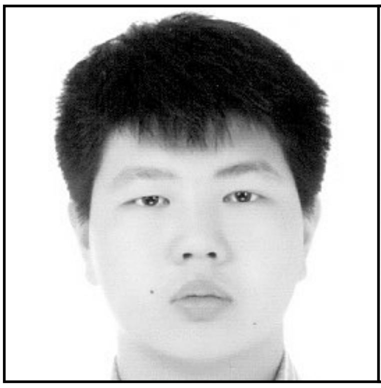


RAPHAEL ADRIAN
GALANG
Manila Science High School

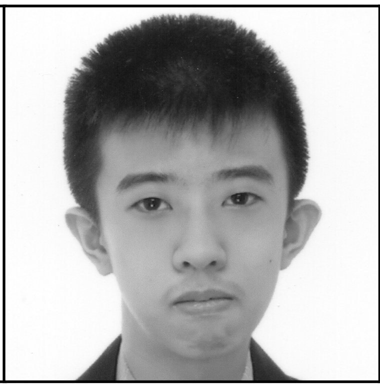
ANDRES RICO
GONZALES III
De La Salle University Integrated School Manila

JUSTIN TENG SOON
KHOO
Regional Science High School - III

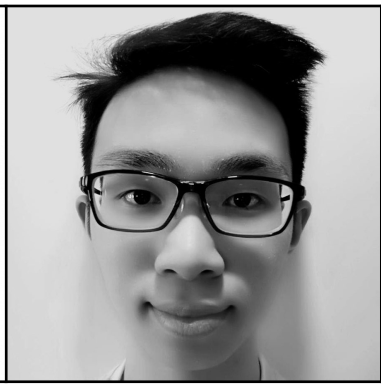
NATIONAL FINALISTS



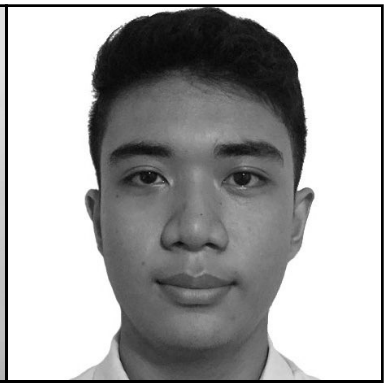
WILLIAM JOSEPH
KING
University of San Carlos - North Campus (STEM)



DARYLL CARLSTEN
KO
Saint Stephen's High School

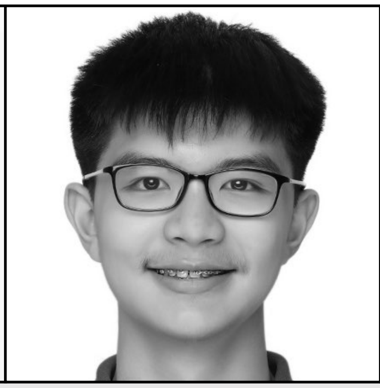


LANCE HEINRICH
LIM
Saint Jude Catholic School

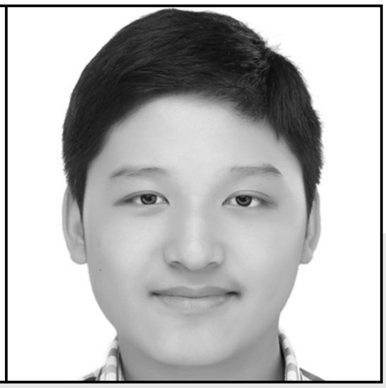


MARKSEN VICTOR
LIZARONDO
Dasmariñas Integrated High School

DION STEPHAN
ONG
Ateneo de Manila Senior High School



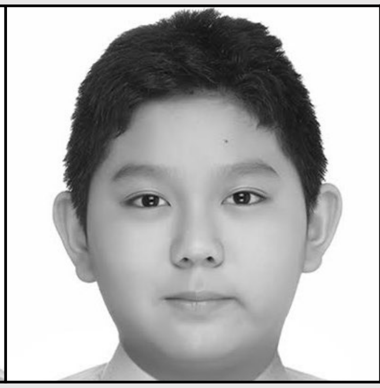
STEVEN
REYES
Saint Jude Catholic School



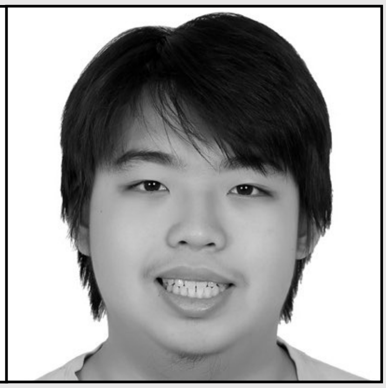
BRYCE AINSLEY
SANCHEZ
Grace Christian College

JUSTIN
SARMIENTO
Marcelo H. Del Pilar National High School

ISSAM
WANG
Manila Science High School



FILBERT EPHRAIM
WU
Victory Christian International School



JONATHAN CONRAD
YU
University of San Carlos - North Campus (STEM)

21st PMO HIGHLIGHTS

22nd Philippine Mathematical Olympiad

Qualifying Stage, 12 October 2019

PART I. Choose the best answer. Each correct answer is worth two points.

1. If $2^{x-1} + 2^{x-2} + 2^{x-3} = \frac{1}{16}$, find 2^x.

(a) $\frac{1}{14}$ (b) $\frac{2}{3}$ (c) $\sqrt[14]{2}$ (d) $\sqrt[3]{4}$

2. If the number of sides of a regular polygon is decreased from 10 to 8, by how much does the measure of each of its interior angles decrease?

(a) $30°$ (b) $18°$ (c) $15°$ (d) $9°$

3. Sylvester has 5 black socks, 7 white socks, 4 brown socks, where each sock can be worn on either foot. If he takes socks randomly and without replacement, how many socks would be needed to guarantee that he has at least one pair of socks of each color?

(a) 13 (b) 14 (c) 15 (d) 16

4. Three dice are simultaneously rolled. What is the probability that the resulting numbers can be arranged to form an arithmetic sequence?

(a) $\frac{1}{18}$ (b) $\frac{11}{36}$ (c) $\frac{7}{36}$ (d) $\frac{1}{6}$

5. Sean and the bases of three buildings A, B, and C are all on level ground. Sean measures the angles of elevation of the tops of buildings A and B to be $62°$ and $57°$, respectively. Meanwhile, on top of building C, CJ spots Sean and determines that the angle of depression of Sean from his location is $31°$. If the distance from Sean to the bases of all three buildings is the same, arrange buildings A, B, and C in order of increasing heights.

(a) C, B, A (b) B, C, A (c) A, C, B (d) A, B, C

6. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) = f(x)/y^2$ for all positive real numbers x and y. Given that $f(25) = 48$, what is $f(100)$?

(a) 1 (b) 2 (c) 3 (d) 4

7. A trapezoid has parallel sides of lengths 10 and 15; its two other sides have lengths 3 and 4. Find its area.

(a) 24 (b) 30 (c) 36 (d) 42

8. Find the radius of the circle tangent to the line $3x + 2y + 4 = 0$ at $(-2, 1)$ and whose center is on the line $x - 8y + 36 = 0$.

(a) $2\sqrt{13}$ (b) $2\sqrt{10}$ (c) $3\sqrt{5}$ (d) $5\sqrt{2}$

9. A circle is inscribed in a rhombus which has a diagonal of length 90 and area 5400. What is the circumference of the circle?

(a) 36π (b) 48π (c) 72π (d) 90π

10. Suppose that n identical promo coupons are to be distributed to a group of people, with no assurance that everyone will get a coupon. If there are 165 more ways to distribute these to four people than there are ways to distribute these to three people, what is n?

(a) 12 (b) 11 (c) 10 (d) 9

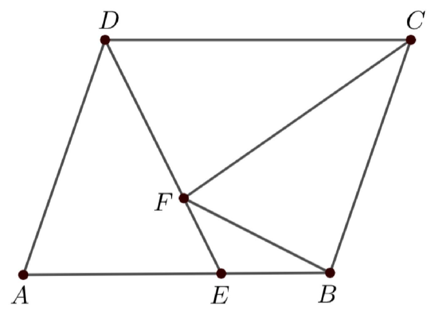
11. Let x and y be positive real numbers such that

$$\log_x 64 + \log_{y^2} 16 = \frac{5}{3} \qquad \text{and} \qquad \log_y 64 + \log_{x^2} 16 = 1.$$

What is the value of $\log_2(xy)$?

(a) 16 (b) 3 (c) $\frac{1}{3}$ (d) $\frac{1}{48}$

12. The figure below shows a parallelogram $ABCD$ with $CD = 18$. Point F lies inside $ABCD$ and lines AB and DF meet at E. If $AE = 12$ and the areas of triangles FEB and FCD are 30 and 162, respectively, find the area of triangle BFC.



(a) 162 (b) 156 (c) 150 (d) 144

13. A *semiprime* is a natural number that is the product of two primes, not necessarily distinct. How many subsets of the set $\{2, 4, 6, \ldots, 18, 20\}$ contain at least one semiprime?

(a) 768 (b) 896 (c) 960 (d) 992

14. The number whose base-b representation is 91_b is divisible by the number whose base-b representation is 19_b. How many possible values of b are there?

(a) 2 (b) 3 (c) 4 (d) 5

15. The number of ordered pairs (a, b) of relatively prime positive integers such that $ab = 36!$ is

(a) 128 (b) 1024 (c) 2048 (d) 4096

PART II. Choose the best answer. Each correct answer is worth three points.

16. Which of the following cannot be the difference between a positive integer and the sum of its digits?

(a) 603 (b) 684 (c) 765 (d) 846

17. Evaluate the sum

$$\sum_{n=0}^{2019} \cos\left(\frac{n^2\pi}{3}\right).$$

(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

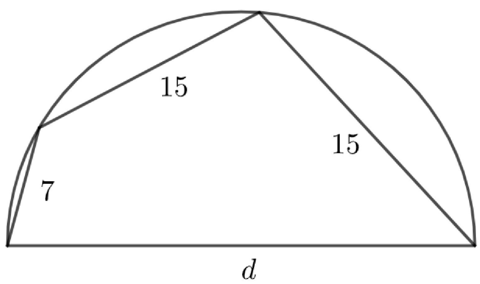
18. There is an unlimited supply of red 4×1 tiles and blue 7×1 tiles. In how many ways can an 80×1 path be covered using nonoverlapping tiles from this supply?

(a) 2381 (b) 3382 (c) 5384 (d) 6765

19. For a real number t, $\lfloor t \rfloor$ is the greatest integer less than or equal to t. How many natural numbers n are there such that $\left\lfloor \frac{n^3}{9} \right\rfloor$ is prime?

(a) 3 (b) 9 (c) 27 (d) infinitely many

20. A quadrilateral with sides of lengths 7, 15, 15, and d is inscribed in a semicircle with diameter d, as shown in the figure below.



Find the value of d.

(a) 18 (b) 22 (c) 24 (d) 25

21. Find the sum of all real numbers b for which all the roots of the equation $x^2 + bx - 3b = 0$ are integers.

(a) 4 (b) -8 (c) -12 (d) -24

22. A number x is selected randomly from the set of all real numbers such that a triangle with side lengths 5, 8, and x may be formed. What is the probability that the area of this triangle is greater than 12?

(a) $\dfrac{3\sqrt{15}-5}{10}$ (b) $\dfrac{3\sqrt{15}-\sqrt{41}}{10}$ (c) $\dfrac{3\sqrt{17}-5}{10}$ (d) $\dfrac{3\sqrt{17}-\sqrt{41}}{10}$

23. Two numbers a and b are chosen randomly from the set $\{1, 2, \ldots, 10\}$ in order, and with replacement. What is the probability that the point (a, b) lies above the graph of $y = ax^3 - bx^2$?

(a) $\dfrac{4}{25}$ (b) $\dfrac{9}{50}$ (c) $\dfrac{19}{100}$ (d) $\dfrac{1}{5}$

24. For a real number t, $\lfloor t \rfloor$ is the greatest integer less than or equal to t. How many integers n are there with $4 \leq n \leq 2019$ such that $\lfloor \sqrt{n} \rfloor$ divides n and $\lfloor \sqrt{n+1} \rfloor$ divides $n + 1$?

(a) 44 (b) 42 (c) 40 (d) 38

25. The number $20^5 + 21$ has two prime factors which are three-digit numbers. Find the sum of these numbers.

(a) 1112 (b) 1092 (c) 1062 (d) 922

PART III. All answers should be in simplest form. Each correct answer is worth six points.

26. Find the number of ordered triples of integers (m, n, k) with $0 < k < 100$ satisfying

$$\frac{1}{2^m} - \frac{1}{2^n} = \frac{3}{k}.$$

27. Triangle ABC has $\angle BAC = 60^\circ$ and circumradius 15. Let O be the circumcenter of ABC and let P be a point inside ABC such that $OP = 3$ and $\angle BPC = 120^\circ$. Determine the area of triangle BPC.

28. A string of 6 digits, each taken from the set $\{0, 1, 2\}$, is to be formed. The string should not contain any of the substrings 012, 120, and 201. How many such 6-digit strings can be formed?

29. Suppose a, b, and c are positive integers less than 11 such that

$$\begin{aligned} 3a + b + c &\equiv abc \pmod{11} \\ a + 3b + c &\equiv 2abc \pmod{11} \\ a + b + 3c &\equiv 4abc \pmod{11} \end{aligned}$$

What is the sum of all the possible values of abc?

30. Find the minimum value of $\dfrac{7x^2 - 2xy + 3y^2}{x^2 - y^2}$ if x and y are positive real numbers such that $x > y$.

Answers

Part I. (2 points each)

1. A
2. D
3. B
4. C
5. A
6. C
7. B
8. A
9. C
10. D
11. A
12. D
13. C
14. B
15. C

Part II. (3 points each)

16. B
17. A
18. C
19. A
20. D
21. D
22. C
23. C
24. B
25. A

Part III. (6 points each)

26. 13
27. $54\sqrt{3}$
28. 492
29. 198
30. $2\sqrt{6} + 2$

22nd Philippine Mathematical Olympiad

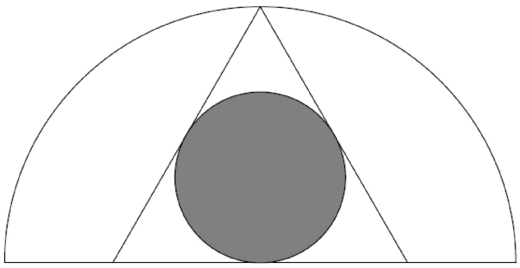
Area Stage, 16 November 2019

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. If the sum of the first 22 terms of an arithmetic progression is 1045 and the sum of the next 22 terms is 2013, find the first term.

2. How many positive divisors do 50,400 and 567,000 have in common?

3. In the figure below, an equilateral triangle of height 1 is inscribed in a semicircle of radius 1. A circle is then inscribed in the triangle. Find the fraction of the semicircle that is shaded.



4. Determine the number of ordered quadruples (a, b, c, d) of odd positive integers that satisfy the equation $a + b + c + d = 30$.

5. Suppose a real number $x > 1$ satisfies

$$\log_{\sqrt[3]{3}}(\log_3 x) + \log_3(\log_{27} x) + \log_{27}(\log_{\sqrt[3]{3}} x) = 1.$$

Compute $\log_3(\log_3 x)$.

6. Let $f(x) = x^2 + 3$. How many positive integers x are there such that x divides $f(f(f(x)))$?

7. In $\triangle XYZ$, let A be a point on (segment) YZ such that XA is perpendicular to YZ. Let M and N be the incenters of triangles XYA and XZA, respectively. If $YZ = 28$, $XA = 24$, and $YA = 10$, what is the length of MN?

8. Find the largest three-digit integer for which the product of its digits is 3 times the sum of its digits.

9. A wooden rectangular brick with dimensions 3 units by a units by b units is painted blue on all six faces and then cut into $3ab$ unit cubes. Exactly $1/8$ of these unit cubes have all their faces unpainted. Given that a and b are positive integers, what is the volume of the brick?

10. In square $ABCD$ with side length 1, E is the midpoint of AB and F is the midpoint of BC. The line segment EC intersects AF and DF at G and H, respectively. Find the area of quadrilateral $AGHD$.

11. A sequence $\{a_n\}_{n\geq 1}$ of positive integers satisfies the recurrence relation $a_{n+1} = n\lfloor a_n/n \rfloor + 1$ for all integers $n \geq 1$. If $a_4 = 34$, find the sum of all the possible values of a_1.

12. Let $(0,0)$, $(10,0)$, $(10,8)$, and $(0,8)$ be the vertices of a rectangle on the Cartesian plane. Two lines with slopes -3 and 3 pass through the rectangle and divide the rectangle into three regions with the same area. If the lines intersect above the rectangle, find the coordinates of their point of intersection.

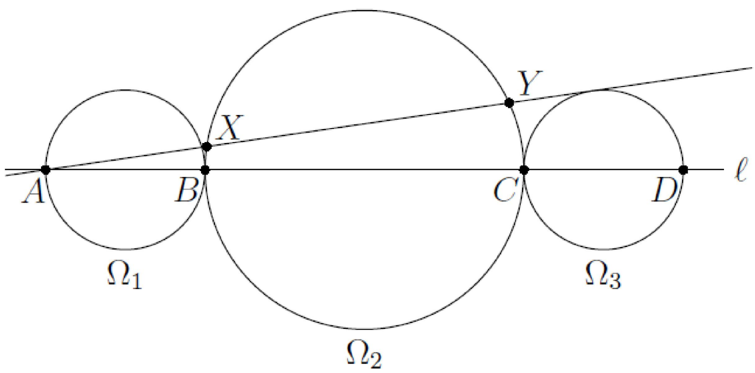
13. For a positive integer x, let $f(x)$ be the last two digits of x. Find $\displaystyle\sum_{n=1}^{2019} f\left(7^{7^n}\right)$.

14. How many positive rational numbers less than 1 can be written in the form $\dfrac{p}{q}$, where p and q are relatively prime integers and $p + q = 2020$?

15. The constant term in the expansion of $\left(ax^2 - \dfrac{1}{x} + \dfrac{1}{x^2}\right)^8$ is $210a^5$. If $a > 0$, find the value of a.

16. Let $A = \{n \in \mathbb{Z} \mid |n| \leq 24\}$. In how many ways can two distinct numbers be chosen (simultaneously) from A such that their product is less than their sum?

17. Points A, B, C, and D lie on a line ℓ in that order, with $AB = CD = 4$ and $BC = 8$. Circles Ω_1, Ω_2, and Ω_3 with diameters AB, BC, and CD, respectively, are drawn. A line through A and tangent to Ω_3 intersects Ω_2 at the two points X and Y. Find the length of XY.



18. A musical performer has three different outfits. In how many ways can she dress up for seven different performances such that each outfit is worn at least once? (Assume that outfits can be washed and dried between performances.)

19. In $\triangle PMO$, $PM = 6\sqrt{3}$, $PO = 12\sqrt{3}$, and S is a point on MO such that PS is the angle bisector of $\angle MPO$. Let T be the reflection of S across PM. If PO is parallel to MT, find the length of OT.

20. A student writes the six complex roots of the equation $z^6 + 2 = 0$ on the blackboard. At every step, he randomly chooses two numbers a and b from the board, erases them, and replaces them with $3ab - 3a - 3b + 4$. At the end of the fifth step, only one number is left. Find the largest possible value of this number.

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. Consider all the subsets of $\{1, 2, 3, \dots, 2018, 2019\}$ having exactly 100 elements. For each subset, take the greatest element. Find the average of all these greatest elements.

2. Let $a_1, a_2, \dots$ be a sequence of integers defined by $a_1 = 3$, $a_2 = 3$, and

$$a_{n+2} = a_{n+1}a_n - a_{n+1} - a_n + 2$$

for all $n \geq 1$. Find the remainder when a_{2020} is divided by 22.

3. In $\triangle ABC$, $AB = AC$. A line parallel to BC meets sides AB and AC at D and E, respectively. The angle bisector of $\angle BAC$ meets the circumcircles of $\triangle ABC$ and $\triangle ADE$ at points X and Y, respectively. Let F and G be the midpoints of BY and XY, respectively. Let T be the intersection of lines CY and DF. Prove that the circumcenter of $\triangle FGT$ lies on line XY.

Answers to the 22nd PMO Area Stage

Part I. (3 points each)

1. $\frac{53}{2}$
2. 72
3. $\frac{2}{9}$
4. 560
5. $\frac{5}{13}$
6. 6
7. $2\sqrt{26}$
8. 951
9. 96
10. $\frac{7}{15}$
11. 130
12. $(5, 9)$
13. 50493
14. 400
15. $\frac{4}{3}$
16. 623
17. $\frac{24\sqrt{5}}{7}$
18. 1806
19. $2\sqrt{183}$
20. 730

Part II. (10 points each, full solutions required)

1. Consider all the subsets of $\{1, 2, 3, \ldots, 2018, 2019\}$ having exactly 100 elements. For each subset, take the greatest element. Find the average of all these greatest elements.

   Solution 1: Let M be the average that we are computing. First, there are $\binom{2019}{100}$ ways to choose a 100-element subset. Next, if x is the largest element, then $x \geq 100$, and there are $\binom{x-1}{99}$ subsets having x as the largest element. Hence

$$M = \frac{\sum_{x=100}^{2019} x\binom{x-1}{99}}{\binom{2019}{100}}.$$

   But note that

$$x\binom{x-1}{99} = 100\binom{x}{100}.$$

   Using this fact and the hockey stick identity, we have

$$\begin{aligned} M &= \frac{100\sum_{x=100}^{2019}\binom{x}{100}}{\binom{2019}{100}} \\ &= \frac{100\binom{2020}{101}}{\binom{2019}{100}} \\ &= \frac{100 \cdot 2020}{101} \\ &= \boxed{2000} \end{aligned}$$

Solution 2: As in Solution 1, the required average M can be written as

$$M = \frac{\sum_{x=100}^{2019} x\binom{x-1}{99}}{\binom{2019}{100}} = \frac{100\binom{99}{99} + 101\binom{100}{99} + \cdots + 2019\binom{2018}{99}}{\binom{2019}{100}}.$$

We can simplify this expression by grouping the terms in the numerator in specific ways, applying the hockey stick identity to each group of terms, and then applying the hockey stick identity again to the resulting terms. This can be done in several ways, two of which are shown next:

Way 2.1: Note that

$$\begin{aligned}
\binom{2019}{100}M &= 100\binom{99}{99} + 101\binom{100}{99} + \cdots + 2019\binom{2018}{99} \\
&= 2020\left[\binom{99}{99} + \binom{100}{99} + \cdots + \binom{2018}{99}\right] - \left[1920\binom{99}{99} + 1919\binom{100}{99} + \cdots + \binom{2018}{99}\right] \\
&= 2020\left[\binom{99}{99} + \binom{100}{99} + \cdots + \binom{2018}{99}\right] \\
&\quad - \left[\binom{99}{99} + \binom{100}{99} + \cdots + \binom{2018}{99}\right] \\
&\quad - \left[\binom{99}{99} + \binom{100}{99} + \cdots + \binom{2017}{99}\right] \\
&\quad - \cdots - \binom{99}{99} \\
&= 2020\binom{2019}{100} - \binom{2019}{100} - \binom{2018}{100} - \cdots - \binom{100}{100} \\
&= 2020\binom{2019}{100} - \binom{2020}{101}.
\end{aligned}$$

where the last two lines follow from the hockey stick identity.

Hence, $M = 2020 - \dfrac{\binom{2020}{101}}{\binom{2019}{100}} = 2020 - \dfrac{2020}{101} = \boxed{2000}$.

Way 2.2: Let

$$X := \binom{99}{99} + \binom{100}{99} + \cdots + \binom{2018}{99} = \binom{2019}{100}.$$

Then

$$\begin{aligned}
\binom{2019}{100}M &= 100\binom{99}{99} + 101\binom{100}{99} + \cdots + 2019\binom{2018}{99} \\
&= 100\left[\binom{99}{99} + \binom{100}{99} + \cdots + \binom{2018}{99}\right] \\
&\quad + \left[\binom{100}{99} + \cdots + \binom{2018}{99}\right] \\
&\quad + \left[\binom{101}{99} + \cdots + \binom{2018}{99}\right] \\
&\quad + \cdots + \binom{2018}{99} \\
&= 100X + \left[X - \binom{100}{100}\right] + \left[X - \binom{101}{100}\right] + \cdots + \left[X - \binom{2018}{100}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
MX &= 100X + 1919X - \left[\binom{100}{100} + \binom{101}{100} + \cdots + \binom{2018}{100}\right] \\
&= 2019X - \binom{2019}{101}
\end{aligned}$$

which gives

$$M = 2019 - \frac{\binom{2019}{101}}{\binom{2019}{100}} = 2020 - \frac{2020}{101} = \boxed{2000}.$$

2. Let $a_1, a_2, \ldots$ be a sequence of integers defined by $a_1 = 3,\ a_2 = 3$, and

$$a_{n+2} = a_{n+1}a_n - a_{n+1} - a_n + 2$$

for all $n \geq 1$. Find the remainder when a_{2020} is divided by 22.

Solution 1: Let $\{F_n\}_{n=1}^{\infty} = \{1, 1, 2, 3, 5, 8, \ldots\}$ be the sequence of Fibonacci numbers. We first claim that $a_n = 2^{F_n} + 1$ for all $n \in \mathbb{N}$. Clearly, this is true for $n = 1, 2$. Let $k \in \mathbb{N}$ and suppose that the claim is true for $n = k$ and for $n = k + 1$. Then

$$\begin{aligned}
a_{k+2} &= a_{k+1}a_k - a_{k+1} - a_k + 2 \\
&= (a_{k+1} - 1)(a_k - 1) + 1 \qquad (1) \\
&= 2^{F_{k+1}}2^{F_k} + 1 \\
&= 2^{F_{k+2}} + 1.
\end{aligned}$$

By strong induction, the claim is proved. Therefore, we now find the remainder when $2^{F_{2020}} + 1$ is divided by 22. To this end, it is easier to find residues modulo 2 and modulo 11 and process them to get the residue modulo 22 (e.g. through Chinese Remainder Theorem).

Clearly, $2^{F_{2020}}$ is even, i.e., $2^{F_{2020}}+1 \equiv 1 \pmod 2$. We will see later that $2^{F_{2020}}+1 \equiv 0 \pmod{11}$. Therefore, by Chinese Remainder Theorem, $\boxed{a_{2020} = 2^{F_{2020}}+1 \equiv 11 \pmod{22}}$.

There are several ways to find the residue of $2^{F_{2020}}+1 \pmod{11}$.

Way 1.1: By Fermat's Little Theorem, $2^{10} \equiv 1 \pmod{11}$. This prompts us to consider the sequence of residues of $F_n \pmod{10}$ in order to find $F_{2020} \pmod{10}$:

$$\{F_n \pmod{10}\}_{n=1}^{\infty} = \{1,1,2,3,5,8,3,1,4,5,9,4,3,7,0,7,7,4,1,5,6,1,7,8,5,3,8,1,9,0,9,9,8,\\ 7,5,2,7,9,6,5,1,6,7,3,0,3,3,6,9,5,4,9,3,2,5,7,2,9,1,0,...cycle\}$$

We see that the sequence is cyclic with period 60. Therefore, since $2020 = 60(33)+40$, we obtain $F_{2020} \equiv 5 \pmod{10}$. Consequently, for some $k \in \mathbb{Z}$,

$$2^{F_{2020}}+1 = 2^{10k+5}+1 \equiv \left(2^{10}\right)^k 2^5 + 1 \equiv 33 \equiv 0 \pmod{11}. \tag{2}$$

Way 1.2: Another way to find $F_{2020} \pmod{10}$ is to get the residues of F_{2020} modulo 2 and 5 and process them to find the residue modulo 10. Again, we list down the sequence of residues modulo 2 and 5: $\{F_n \pmod 2\}_{n=1}^{\infty} = \{1,1,0,...cycle\}$ which has period 3, and

$$\{F_n \pmod 5\}_{n=1}^{\infty} = \{1,1,2,3,0,3,3,1,4,0,4,4,3,2,0,2,2,4,1,0,...cycle\},$$

which has period 20. Since, $2020 \equiv 1 \pmod 3$ and $2020 \equiv 0 \pmod{20}$, then $F_{2020} \equiv 1 \pmod 2$ and $F_{2020} \equiv 0 \pmod 5$. Therefore, $F_{2020} \equiv 5 \pmod{10}$ by Chinese Remainder Theorem. Thus, (2) holds.

Solution 2: Inspired by the factorization in (1), we define $b_n = a_n - 1$ for all $n \in \mathbb{N}$. Then $b_1 = b_2 = 2$ and (1) simplifies to

$$b_{n+2} = b_{n+1}b_n. \tag{3}$$

We observe that $b_1 = 2^{F_1}$, $b_2 = 2^{F_2}$, and (3) implies $b_3 = 2^{F_3}$. This pattern continues and obviously shows that $b_n = 2^{F_n}$ for all $n \in \mathbb{N}$. Very similar arguments to Solution 1 will give $b_{2020} \equiv 0 \pmod 2$ and $b_{2020} \equiv 10 \pmod{11}$. Therefore, by Chinese Remainder Theorem, $b_{2020} \equiv 10 \pmod{22}$. Equivalently, $a_{2020} \equiv 11 \pmod{22}$.

Solution 3: By bashing, we can list down the residues of $\{a_n\}_{n=1}^{\infty}$ (or of $\{b_n\}_{n=1}^{\infty}$ as defined in Solution 2):

$$\{a_n \pmod{22}\}_{n=1}^{\infty} = \{3,3,5,9,11,15,9,3,17,11,7,17,9,19,13,19,19,17,3,11,21,3,19,15,\\ 11,9,15,3,7,13,7,7,15,19,11,5,19,7,21,11,3,21,19,9,13,9,9,21,7,11,\\ 17,7,9,5,11,19,5,7,3,13,...cycle\}.$$

Since the sequence is cyclic with period 60, and $2020 \equiv 40 \pmod{60}$, then $a_{2020} \equiv 11 \pmod{22}$.

3. In $\triangle ABC$, $AB = AC$. A line parallel to BC meets sides AB and AC at D and E, respectively. The angle bisector of $\angle BAC$ meets the circumcircle of $\triangle ABC$ and $\triangle ADE$ at points X and Y, respectively. Let F and G be the midpoints of BY and XY, respectively. Let T be the intersection of lines CY and DF. Prove that the circumcenter of $\triangle FGT$ lies on line XY.

Solution 1: Let F' be the reflection of F over line XY. Observe that, by symmetry, we get $\angle ADY = \angle AEY$. As quadrilateral $ADYE$ is cyclic, both angles must be right, and hence $\angle BDY$ is right as well.

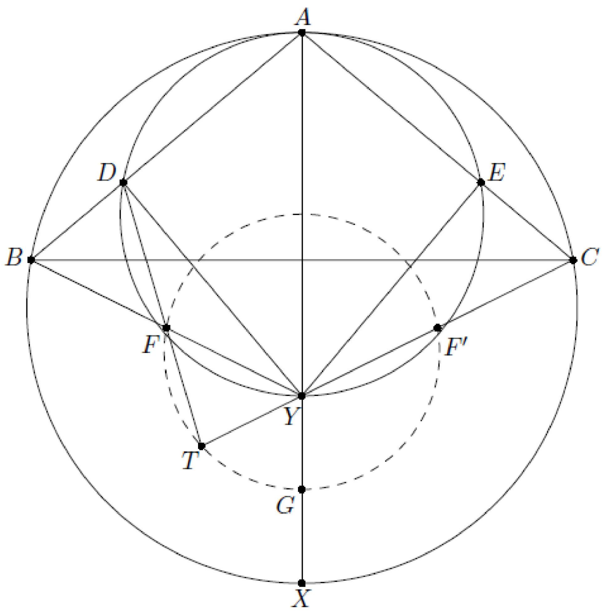
Thus F is the circumcenter of triangle BDY, so $\angle FDB = \angle FBD$. By reflecting over XY, we also get $\angle FBD = \angle F'CE$. Thus

$$\measuredangle TCA = \measuredangle F'CE = \measuredangle DBF = \measuredangle FDB = \measuredangle TDA,$$

so D, A, T, and C are concyclic. This implies that

$$\measuredangle FTF' = \measuredangle DTC = \measuredangle DAC = \measuredangle BAC = \measuredangle BXC = \measuredangle FGF',$$

the last step following from $FG \parallel BX$ and $F'G \parallel CX$. Thus the points F, T, G, and F' are concyclic. Their circumcenter must lie on the perpendicular bisector of FF', which is line XY. But this is also the circumcenter of triangle FGT, as desired.



Solution 2: As in Solution 1, AY and AX are diameters. It follows that

$$\angle ADY = \angle ABX = 90^\circ \implies BX \perp BD \perp DY.$$

Hence $BX \parallel DY$. As FG is a midline of $\triangle BXY$, it follows that it is the perpendicular bisector of BD. Then

$$\measuredangle GFT = \measuredangle GFD = \measuredangle BFG = \measuredangle YFG = \measuredangle GF'Y = \measuredangle GF'T,$$

which implies that $FF'GT$ is cyclic. The logic in Solution 1 finishes the proof.

Remark: The directed angles are necessary here due to configuration issues.

MSP Annual Convention 2019, General Santos City.

HARI
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FOUNDATION

HARI Foundation, Inc. Celebrates the Power of the Heart to Move Lives

From its humble beginnings over a decade ago, Hyundai Asia Resources, Inc. (HARI), the official Philippine distributor of Hyundai vehicles, has earned its place as a leader in the automotive industry. More than just reaping success as a business, HARI has taken upon itself to be a catalyst of change in Philippine society, inspired by the global vision of Hyundai Motor Company (HMC) to build a better world for all.

Through its Corporate Social Responsibility (CSR) arm, HARI Foundation, Inc. (HFI), HARI has engaged employees, dealerships, and customers, as well as partners in the public and private sectors in trail-blazing endeavors across the country in the areas of education, environment, community development, healthcare, and women's and children's rights.

Among HFI's roster of partners in nation building are Gawad Kalinga, St. Scholastica's Priory, the Department of Science and Technology (DOST), National Museum, The Mind Museum, Synergeia Foundation, the ASEAN Center for Biodiversity, and UP Philippine General Hospital.

Fully aware that poverty is more than just the lack of resources, but of options to pursue a better life, HFI has streamlined its efforts to a more targeted, long-term, multi-disciplinary, and multi-stakeholder solution. Education is at the core of HFI's contribution to the Philippine agenda for sustainability.

Says HFI President Ma. Fe Perez-Agudo, "I have always been a believer in empowering people through education. By engaging with like-minded partners from the government and the private sectors, we aim to be the Filipino's lifetime partner in building a sustainable, empowered nation one community at a time."

HFI's over 10 years of working with and learning from its various CSR partners and beneficiaries have led to its developing programs that respond to the needs of our times.

The Hyundai New Thinkers Circuit (HNTC), designed in partnership with the DOST, served as an innovative science literacy program that provokes, fosters, and nurtures leadership and the innovative spirit among outstanding high school students who can take the lead in building a climate change-resilient Philippines. Launched in 2013, HNTC yielded 11 scholars who are pursuing studies in the sciences at the country's top universities.

On March 22, 2017, HFI, Hyundai Motor Company (HMC) Korea, the Institute for Global Education, Exchange and Internship (IGEEI), and the Tanay local government collaborated to build the pilot Rain Water Harvesting System in Rawang Elementary School in Tanay City. This filtration method, an invention of Prof. Han Moo Young of Seoul National University, is capable of producing and storing potable water from rain gathered in roof gutters. Some 200 students of Rawang Elementary School were among its first beneficiaries.This project, which will soon be replicated in other under-served communities in the country, bagged for HFI the 2017 Gold Award from the Society of Philippine Motoring Journalists (SPMJ).

HFI has likewise been an active supporter of Gawad Kalinga (GK) efforts at rehabilitating war-torn Marawi through the donation of vehicles that ply the province as Kusina ng Kalinga (KNK) vans. KNK is GK's platform to address malnutrition among children in public schools, on the streets, and in conflict areas. HFI is taking a step further to help Marawi back to its feet with the rebuilding of public schools in strategic areas of the province.

Finally, HFI in partnership with UP-PGH and the UP-PGH Cancer Institute, aims to empower women through proactive healthcare. Hyundai in the Philippines, through HFI, has donated a Hyundai H350 luxury van customized into a state-of-the-art mobile cancer diagnostic clinic that will transport UP-PGH medical missions to under-served sectors of the country. Dubbed the Alagang Breastfriend project, this comprehensive breast cancer awareness campaign is envisioned to provide women information and access to important resources and technology that would improve their overall well-being, thereby enabling them to lead healthier, more productive lives.

"In a nutshell, the story of HFI in this new century goes by the acronym H.E.A.R.T.—Health, Education, Arts, Rebuilding, and Transformative Leadership. We kick off a new leg in our journey to broaden our reach and design programs that are more meaningful to people, especially at the grassroots", concluded Ms. Agudo.

More than a foundation, HFI represents a movement to build a better world for all, one community at a time, driven by faith in the power of the Filipino heart to innovate, engage, and collaborate to drive shared dreams forward and give rise to generations of leaders capable of advancing sustainability in all important aspects of our lives.

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