PHILIPPINE MATHEMATICAL OLYMPIAD

MCMLXXXIV

PMO

21

21

NATIONAL STAGE

University of the Philippines Diliman

26 January 2019

SCHEDULE

7:00 AM - 7:30 AM	**Registration** UP NISMED
7:30 AM - 12:00 NN	**Written Phase** UP NISMED
12:00 NN - 1:00 PM	**Lunch Break**
1:00 PM - 5:00 PM	**Oral Phase** UP NISMED Auditorium
6:30 PM - 9:00 PM	**Awarding Ceremonies** UP SOLAIR Auditorium

ABOUT THE PMO

First held in **1984**, the PMO was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are: **(1)** to awaken greater interest in and promote the appreciation of mathematics among students and teachers; **(2)** to identify mathematically-gifted students and motivate them towards the development of their mathematical skills; **(3)** to provide a vehicle for the professional growth of teachers; and **(4)** to encourage the involvement of both public and private sectors in the promotion and development of mathematics education in the Philippines.

The PMO is the first part of the selection process leading to participation in the **International Mathematical Olympiad (IMO).** It is followed by the **Mathematical Olympiad Summer Camp (MOSC)**, a five-phase program for the twenty two national finalists of PMO. The four selection tests given during the second phase of MOSC determine the tentative Philippine Team to the IMO. The final team is determined after the third phase of MOSC.

The PMO this year is the twenty-first since 1984. four thousand five hundred seventeen (4517) junior and senior high school students from about four hundred schools all over the country took the qualifying examination, out of these, two hundred twenty-two (222) students made it to the Area Stage. Now, in the National Stage, the number is down to twenty-two (22) and these twenty-two students will compete for the top three positions and hopefully move on to represent the country in the **60th International Mathematical Olympiad**, which will be held in **Bath, United Kingdom**, from **11 to 22 July 2019**.

MESSAGE

Happy New Year!

It has been 21 years since the country's best talent pooling in mathematics was established through the Philippine Mathematical Olympiad (PMO). In such a span it has produced some of the brightest thinkers in the country who are now making their marks from everywhere in the globe. This legacy of the PMO is what inspires us from the science community to maintain our support to this productive platform.

The PMO, as the longest running local mathematics competition, continues to hone talents in the field and we believe that it's only getting better and better. This year's finalists are out to prove themselves as the best in their respective classes, provinces, and regions, and at this final stage, they all vie for the top spots. Looking back in the past decade, it excites us to know that we have been rising consistently and the world has taken notice. With three gold medals from the International Mathematical Olympiad (IMO) already in our trophy cabinets, we remain hoping for more. We are certainly confident that with the PMO, we are training the next breed of IMO medalists very well.

With tight competition, we are bound to produce not just great competitors or future mathematicians; we are bound to create more leaders that will help us propel this country to better days. We call on our finalists to follow in the footsteps of the alumni of the PMO to be outstanding professionals in their respective fields, and may your journeys begin here. Together with the Mathematics Society of the Philippines and the schools taking part in this competition, the Science Education Institute (SEI) shall remain as a major driver of development through science and technology.

We wish our participants all the best.

JOSETTE T. BIYO
Director
Science Education Institute
Department of Science and Technology

MESSAGE

The Mathematical Society of the Philippines (MSP) is committed to the promotion and development of mathematics research and mathematics education in the country. It aims to raise national awareness of the importance of developing young mathematical talents who will later on become leaders in their chosen fields as mathematicians, engineers and scientists and who will contribute to nation building. Part of the MSP's initiative towards this goal is to identify and nurture young mathematical talents through the Philippine Mathematical Olympiad (PMO). Through the successful partnership between the MSP and the Department of Science and technology (DOST), the PMO has become an important part in the selection of the Philippine Team into the International Mathematical Olympiad (IMO). Over the years, we have seen our IMO Teams' improvement in performance at the IMO. Over the last three years, the teams have consistently won medals and honorable mentions. They have surely made our country proud! Thanks to all the team leaders, deputy leaders and trainers who have contributed and helped in taking us to where we are now.

It is then with great pleasure and enthusiasm that I would like to congratulate all winners of the area stage and all the twenty-two national stage finalists of the 21st PMO. Out of 4,517 students who joined the qualifying stage, you have made it this far. Congratulations!

To the twenty-two finalists, you are now faced with the challenge and the opportunity to get into the IMO team. As you train at the Math Olympiad Summer Camp (MOSC), it is my hope that you grow and learn through the experience of training with other talented minds.

I will not forget to thank the dedicated PMO Team led by Dr. Nerissa M. Abara. It is never easy to organize a big event such as PMO but they have done a very good job in working out all the details and in coordinating to make everything work out fine. Thanks to all regional coordinators, coaches, school administrators, parents and sponsors. You have all made the 21st PMO a big success!

On behalf of the MSP National Board of Directors, I would like to greet and welcome everyone to the 21st PMO National Finals!

EMMANUEL A. CABRAL

President

Mathematical Society of the Philippines

MESSAGE

Congratulations to this year's winners of the Philippine Mathematical Olympiad (PMO).

Hyundai and HARI Foundation, Inc. (HFI) are proud partners in the training and preparation of outstanding Filipino talent competing for top honors at the International Mathematical Olympiad (IMO).

We, together with your parents and coaches, look forward to another winning year.

However, apart from the thrill of bringing home the Gold, we want to see you put your genius, idealism and abilities at the service of our country by involving yourselves in the country's science and technology programs.

Thus, we unleash the power of numbers to build a better country and a better world for all.

Gusto natin, Kaya natin, Sama tayo tungo sa tagumpay!

Mabuhay, team Philippines!

MA. FE PEREZ-AGUDO
President
HARI Foundation, Inc.

MESSAGE

Manulife
BUSINESS PROCESSING SERVICES

My warmest greetings and congratulations to the area stage winners and national finalists of the 21st Philippine Mathematical Olympiad.

As preparations for the next International Mathematical Olympiad begins, I wish to express my gratitude to the coaches, trainers and parents of the participants. Your all-out support is one of the reasons why we have such talented students.

To the Philippine Team, good luck and I wish you all the best! Your hard work and commitment to your craft is truly an inspiration not only to the MBPS community but to the rest of the country.

BOB BUIAROSKI
Head
Manulife Business Processing Services, Inc.

MESSAGE

Congratulations to the Philippine Mathematical Olympiad National Stage contestants and winners—you have come a long way to engage in mathematics, the poetry of logic, to quote Einstein! Having hurdled the different stages, you are living examples of how the discipline has made those who engage in it to be methodical and patient. With the analytical skills and speed and accuracy that mathematics has developed in you, you will undoubtedly thrive in this age of mathematically-driven world and innovations. I urge you to continue with your passion in mathematics and one day, it will not be a surprise to see you come up with surprising connections that will benefit us all.

Congratulations, too, to the mentors! In nurturing your student's interest and developing their skills further, you have contributed to their ability to go about daily living using a most important tool. You are an important link to a significant segment if our present young citizens and future leaders of the country.

I salute each of you, mentors and mentees, and I wish the Philippine Mathematical Olympiad more power!

LUCIO C. TAN
Vice-Chairman, Board of Trustees
Foundation for Upgrading the
Standard of Education Inc.

MESSAGE

On behalf of Sharp Calculators and Collins International Trading Corporation, our warmest greetings to the Mathematical Society of the Philippines on your 21st Philippine Mathematical Olympiad.

It's a great honor for Sharp Calculators to be a part of this prestigious mathematics competition, and we are one with MSP's goal in striving for the quality education and academic excellence to our Filipino youth.

We are in full support with the Mathematical Society of the Philippine's advocacy in helping the Filipino students to advance more in the field of mathematics to be able to keep pace in global standards, and to bring pride and honor to our country, the Philippines.

"Success usually comes to those who are too busy to be looking for it."

-- Henry David Thoreau

To all the people behind MSP, Congratulations. We in Sharp Calculator, assure you that we are always at your back to support all your programs in striving quality in Mathematics.

God bless and more power!

JOHN DE JESUS
National Sales and Marketing Manager
Collins International Trading Corporation
Sharp Calculators Philippines

MESSAGE

My warmest greetings and congratulations to the Mathematical Society of the Philippines (MSP) on the 21st year of the Philippine Mathematical Olympiad (PMO). We also congratulate this year's 222 area stage winners and 22 national finalists. May these achievers continue to uphold excellence and inspire the academic community—students, teachers, and administrators—to work harder in uplifting the quality of Philippine education.

C & E Publishing, Inc. has always taken pride in being a partner of the MSP and the PMO in promoting a greater interest in and appreciation of mathematics education among students and teachers. As a leading developer, publisher, and provider of globally competitive educational resources, we are grateful for the opportunity to work with you in meeting the needs of 21st-century learners and educators and fostering academic and professional excellence.

In line with this, we commend the PMO for helping young Filipino mathematicians in honing their skills and reaching their full potential. May you continue to build a dynamic community of math enthusiasts who will bring honor and glory to the country.

To the national finalists who will compete for the top three spots to represent the country in the upcoming 60th International Mathematical Olympiad, I wish you the best of luck. May the Philippine team emerge victorious.

JOHN EMYL G. EUGENIO
Chief Operating Officer
C&E Publishing, Inc.

THE PMO TEAM

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ASSISTANT DIRECTORS
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Melvin Vidar
Jasmin-Mae Santos

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NCR
Renier Mendoza
Jesus Paolo Joven
Engel John dela Vega

AREA STAGE WINNERS

LUZON

1	**Bona, Sarji Elijah**	Palawan Hope Christian School
2	**Chan, Enzo Rafael**	Bayanihan Institute
3	**Lizarondo, Marksen Viktor**	Congressional Integrated High School

VISAYAS

1	**Yu, Jonathan Conrad**	University of San Carlos – Talamban
2	**Tamayo, Annika Angela Mei**	Ateneo de Iloilo – SMCS
2	**Yap, Christopher James**	St. John's Institute

MINDANAO

1	**Ty, Sean Anderson**	Zamboanga Chong Hua High School
2	**Ty, Stephen James**	Zamboanga Chong Hua High School
3	**Lim, Fedrick Lance**	Zamboanga Chong Hua High School

NCR

1	**Chua, Eion Nikolai**	International School Manila
2	**Sanchez, Bryce Ainsley**	Grace Christian College
3	**Dela Cruz, Vincent**	Valenzuela City School of Mathematics and Science
3	**Reyes, Steven**	Saint Jude Catholic School

PRIZES

The prizes for the TOP THREE for each AREA (Luzon, Visayas, Mindanao, NCR) are:

FIRST PLACE - Medal and SHARP Calculator

SECOND PLACE - Medal and SHARP Calculator

THIRD PLACE - Medal and SHARP Calculator

The prizes for the TOP THREE in the NATIONAL STAGE are:

CHAMPION - P 20,000, Medal, Trophy, and SHARP Calculator

FIRST RUNNER-UP - P 15,000, Medal, Trophy, and SHARP Calculator

SECOND RUNNER-UP - P 10,000, Medal, Trophy, and SHARP Calculator

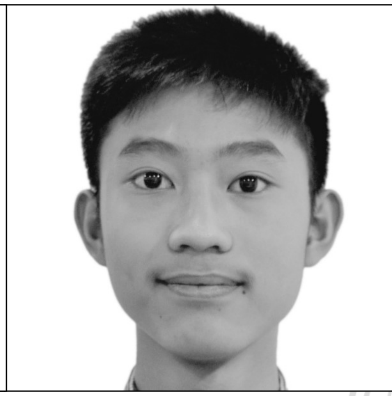
The coaches of the top three will receive P 5,000, P 3,000, and P 2,000, respectively.

The schools of the top three will receive trophies.

Each of the National Finalists will receive a medal and SHARP goodies.

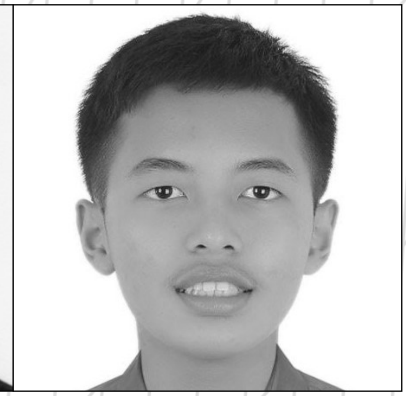
NATIONAL FINALISTS

AERAM CLESTER
ALBO
De La Salle University Integrated School - Manila

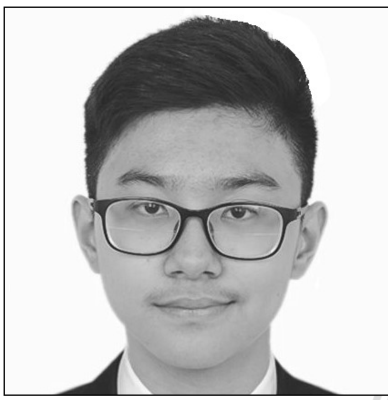


IMMANUEL JOSIAH
BALETE
St. Stephen’s High School

DAN ALDEN
BATERISNA
De La Salle University Integrated School - Manila

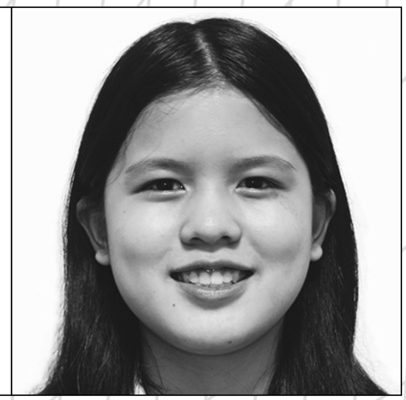


DOMINIC LAWRENCE
BERMUDEZ
Philippine Science High School - Main Campus

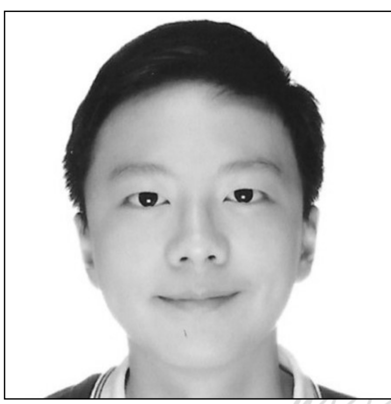


SARJI ELIJAH
BONA
Palawan Hope Christian School

ENZO RAFAEL
CHAN
Bayanihan Institute

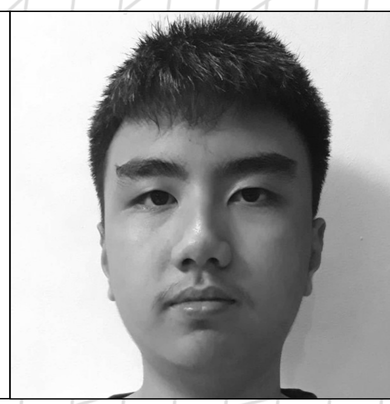


JINGER
CHONG
Saint Jude Catholic School

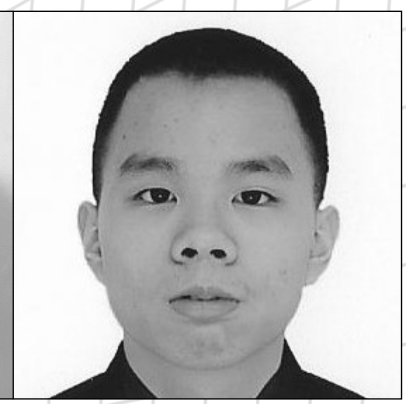


EION NIKOLAI
CHUA
International School Manila

MADELYN ESTHER
CRUZ
Philippine Science High School - Main Campus

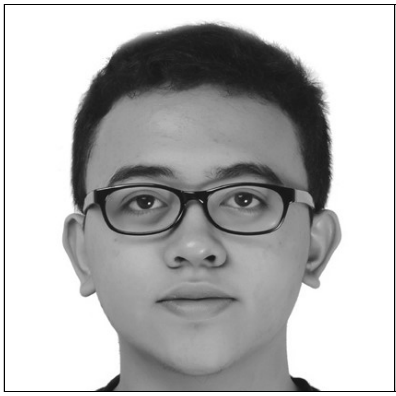


VINCENT
DELA CRUZ
Valenzuela City School of Mathematics and Science

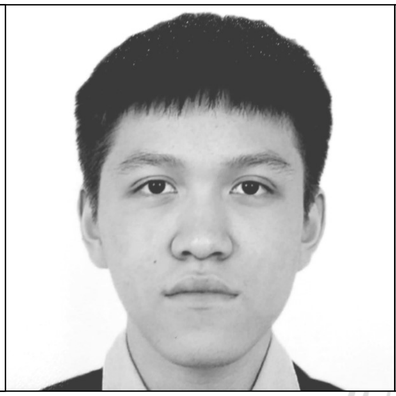


LAWRENCE GABRIEL
DY
CCF Life Academy Foundation Inc.

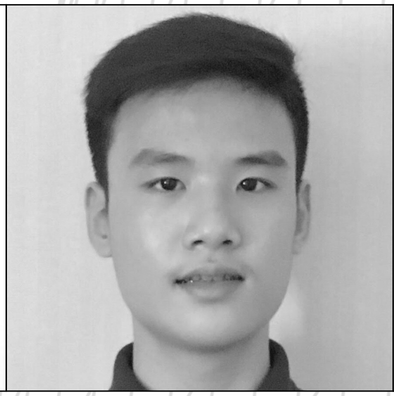
NATIONAL FINALISTS



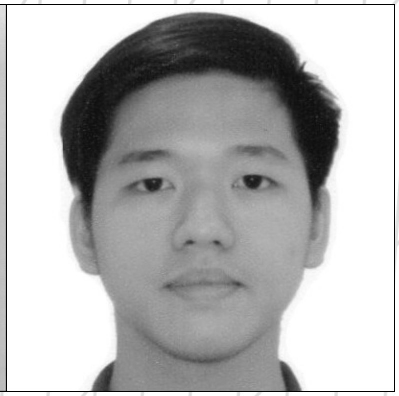
CHRISTIAN PHILIP
GELERA
Philippine Science High School - Main Campus



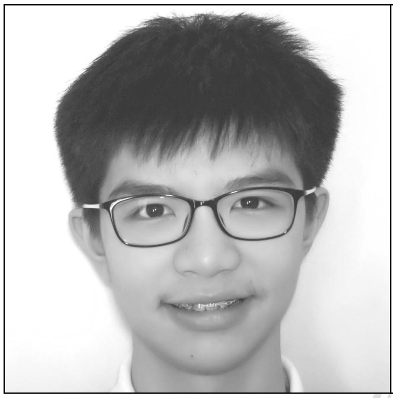
LANCE ADRIAN
KO
St. Stephen’s High School



DION STEPHAN
ONG
Ateneo de Manila Senior High School

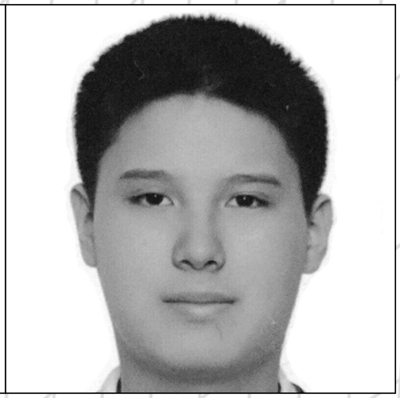


GABRIEL JOSEPH
PUA
St. Stephen’s High School



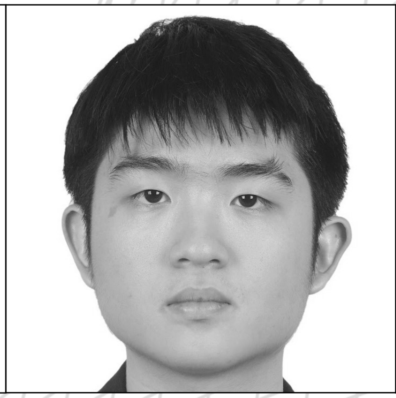
STEVEN
REYES
Saint Jude Catholic School

BRYCE AINSLEY
SANCHEZ
Grace Christian College



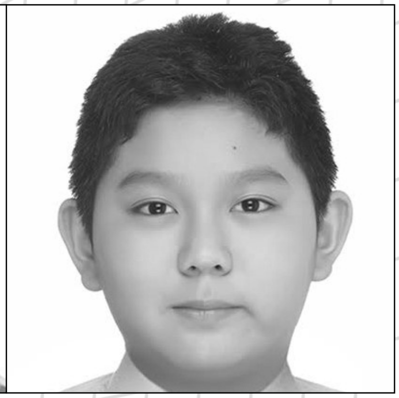
RYAN MARK
SHAO
Xavier School

GWYNETH MARGAUX
TANGOG
Southville International School and Colleges



SEAN ANDERSON
TY
Zamboanga Chong Hua High School

ISSAM
WANG
Manila Science High School



FILBERT EPHRAIM
WU
Victory Christian International School

20TH PMO HIGHLIGHTS

21st Philippine Mathematical Olympiad

Qualifying Stage, 20 October 2018

PART I. Choose the best answer. Each correct answer is worth two points.

1. The measures of the angles of a pentagon form an arithmetic sequence with common difference 15°. Find the measure of the largest angle.

   (a) 78° (b) 103° (c) 138° (d) 153°

2. If $x - y = 4$ and $x^2 + y^2 = 5$, find the value of $x^3 - y^3$.

   (a) -24 (b) -2 (c) 2 (d) 8

3. Five numbers are inserted between 4 and 2916 so that the resulting seven numbers form a geometric sequence. What is the the fifth term of this geometric sequence?

   (a) 324 (b) 416 (c) 584 (d) 972

4. The constant term in the expansion of $\left(3x^2 - \dfrac{1}{x}\right)^6$ is

   (a) 189 (b) 135 (c) 90 (d) 54

5. Juan has 4 distinct jars and a certain number of identical balls. The number of ways that he can distribute the balls into the jars such that each jar has at least one ball is 56. How many balls does he have?

   (a) 9 (b) 8 (c) 7 (d) 6

6. A regular octagon of area 48 is inscribed in a circle. If a regular hexagon is inscribed in the same circle, what would its area be?

   (a) $12\sqrt{10}$ (b) $18\sqrt{6}$ (c) $24\sqrt{3}$ (d) $30\sqrt{2}$

7. What is the smallest positive integer which when multiplied to $24^4 + 64$ makes the product a perfect square?

   (a) 1037 (b) 2074 (c) 5185 (d) 10370

8. A bowl of negligible thickness is in the shape of a truncated circular cone, with height 4 in and upper and lower radii of 9 in and 6 in, respectively. What is the volume of the bowl?

   (a) 276π in^3 (b) 248π in^3 (c) 234π in^3 (d) 228π in^3

9. A circle is tangent to the line $2x - y + 1 = 0$ at the point $(2, 5)$ and the center is on the line $x + y - 9 = 0$. Find the radius of the circle.

(a) $\sqrt{14}$ (b) 4 (c) $3\sqrt{2}$ (d) $2\sqrt{5}$

10. Suppose that 16 points are drawn on a plane such that exactly 7 of these points are collinear. Any set of three points which do not all belong to the 7 are noncollinear. If 3 random points are selected from the 16 points, what is the probability that a triangle can be formed by joining these points?

(a) $\frac{15}{16}$ (b) $\frac{17}{20}$ (c) $\frac{19}{20}$ (d) $\frac{63}{80}$

11. The points $(0, -1)$, $(1, 1)$, and (a, b) are distinct collinear points on the graph of $y^2 = x^3 - x + 1$. Find $a + b$.

(a) -6 (b) -2 (c) 1 (d) 8

12. What is the probability that a positive divisor of $2^{20}3^{17}$ also divides $2^8 3^6$?

(a) $\frac{12}{85}$ (b) $\frac{2}{9}$ (c) $\frac{1}{9}$ (d) $\frac{1}{6}$

13. Let ABC be a right triangle where $AB = 7$, $BC = 24$, and with hypotenuse AC. Point D is on AC such that $AD : DC = 2 : 3$. Let m and n be the relatively prime positive integers such that $BD^2 = \frac{m}{n}$. What is $m + n$?

(a) 554 (b) 550 (c) 544 (d) 540

14. In chess, a knight moves by initially taking two steps in any horizontal or vertical direction and then taking one more step in any direction that is perpendicular to its initial movement. Suppose Renzo places a knight on a random tile on an 8×8 chessboard. Find the probability that he can land on a corner tile in exactly two moves.

(a) $\frac{3}{16}$ (b) $\frac{1}{4}$ (c) $\frac{9}{16}$ (d) $\frac{5}{8}$

15. In rectangle $ABCD$, point Q lies on side AB such that $AQ : QB = 1 : 2$. Ray CQ is extended past Q to R so that AR is parallel to BD. If the area of triangle ARQ is 4, what is the area of rectangle $ABCD$?

(a) 108 (b) 120 (c) 132 (d) 144

PART II. Choose the best answer. Each correct answer is worth three points.

1. How many two-digit numbers are there such that the product of their digits is equal to a prime raised to a positive integer exponent?

(a) 27 (b) 28 (c) 29 (d) 30

2. $ABCDEF$ is a six-digit perfect square in base 10 such that $DEF = 8 \times ABC$. What is $A+B+C+D+E+F$? (Note that $ABCDEF$, ABC, and DEF should be interpreted as numerals in base 10 and not as products of digits.)

(a) 18 (b) 27 (c) 36 (d) 45

3. Quadrilateral $ABCD$ has $AB = 25$, $BC = 60$, $CD = 39$, $DA = 52$, and $AC = 65$. What is the inradius of $\triangle BCD$?

(a) 14 (b) 15 (c) 16 (d) 18

4. Find the sum of the first 20 positive integers that are multiples of either 3 or 7 but not both.

(a) 336 (b) 399 (c) 529 (d) 592

5. Q is a rational function with $xQ(x+2018) = (x-2018)Q(x)$ for all $x \notin \{0, 2018\}$. If $Q(1) = 1$, what is $Q(2017)$?

(a) 2020 (b) 2019 (c) 2018 (d) 2017

6. How many ordered pairs (x, y) of positive integers are there such that $1 \leq x \leq y \leq 20$ and both $\frac{y}{x}$ and $\frac{y+2}{x+2}$ are integers?

(a) 38 (b) 36 (c) 34 (d) 32

7. An entertainment agency has seven trainees. Each of the trainees does at least one of dancing, singing, and rapping, and no two trainees have the same skill set. How many ways can the agency choose three trainees to form a group, provided that the group must have at least one dancer, one singer, and one rapper (who are not necessarily distinct)?

(a) 26 (b) 29 (c) 32 (d) 35

8. Let $\triangle ABC$ be a right triangle such that its hypotenuse AC has length 10 and $\angle BAC = 15°$. Let O be the center of the circumcircle of $\triangle ABC$. Let E be the point of intersection of the lines tangent to the circumcircle at points A and B, and F be the point of intersection of the lines tangent to the circumcircle at points B and C. The area of $\triangle OEF$ is

(a) $\frac{25}{8}$ (b) 50 (c) 100 (d) $\frac{225}{8}$

9. A real number x is chosen randomly from the interval $(0, 1)$. What is the probability that $\lfloor \log_5(3x) \rfloor = \lfloor \log_5 x \rfloor$? (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

(a) $\frac{1}{6}$ (b) $\frac{5}{36}$ (c) $\frac{1}{9}$ (d) $\frac{1}{12}$

10. What is the remainder when $1^{2018} + 2^{2018} + \cdots + 2017^{2018}$ is divided by 2018?

(a) 0 (b) 2 (c) 1009 (d) 2017

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. Suppose two numbers are randomly selected in order, and without replacement, from the set $\{1, 2, 3, \ldots, 888\}$. Find the probability that the difference of their squares is not divisible by 8.

2. Let $P(x)$ be a polynomial with degree 2018 whose leading coefficient is 1. If $P(n) = 3n$ for $n = 1, 2, \ldots, 2018$, find $P(-1)$.

3. A sequence $\{a_n\}_{n\geq 1}$ of positive integers is defined by $a_1 = 2$ and for integers $n > 1$,

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} + \frac{n}{a_n} = 1.$$

   Determine the value of $\displaystyle\sum_{k=1}^{\infty} \frac{3^k(k+3)}{4^k a_k}$.

4. In triangle ABC, D and E are points on sides AB and AC respectively, such that BE is perpendicular to CD. Let X be a point inside the triangle such that $\angle XBC = \angle EBA$ and $\angle XCB = \angle DCA$. If $\angle A = 54°$, what is the measure of $\angle EXD$?

5. Define $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ and $h : \mathbb{R} \setminus \{\sqrt{3}\} \to \mathbb{R}$ as follows:

$$g(x) = \frac{1+x}{1-x} \qquad \text{and} \qquad h(x) = \frac{\sqrt{3}+3x}{3-\sqrt{3}x}.$$

   How many ways are there to choose $f_1, f_2, f_3, f_4, f_5 \in \{g, h\}$, not necessarily distinct, such that $(f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5)(0)$ is well-defined and equal to 0?

Answers

Part I. (2 points each)

1. C
2. B
3. A
4. B
5. A
6. B
7. C
8. D
9. D
10. A
11. D
12. D
13. A
14. D
15. B

Part II. (3 points each)

1. C
2. B
3. A
4. C
5. D
6. D
7. C
8. B
9. A
10. C

Part III. (6 points each)

1. $\frac{5}{8}$
2. $2019! - 3$
3. $\frac{21}{8}$
4. 36°
5. 8

21st Philippine Mathematical Olympiad

Area Stage, 24 November 2018

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. How many distinct prime factors does $5^{14} - 30 + 5^{13}$ have?

2. Given that a and b are real numbers satisfying the equation

$$\log_{16} 3 + 2\log_{16}(a-b) = \frac{1}{2} + \log_{16} a + \log_{16} b,$$

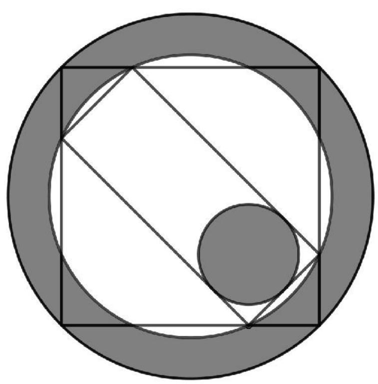
find all possible values of $\frac{a}{b}$.

3. Find the minimum value of the expression $\sqrt{(x-1)^2+(y+1)^2} + \sqrt{(x+3)^2+(y-2)^2}$.

4. In how many ways can the letters of the word COMBINATORICS be arranged so that the letters C, O, A, C, T, O, R, S appear in that order in the arrangement (although there may be letters in between)?

5. Let N be the smallest positive integer divisible by 20, 18, and 2018. How many positive integers are both less than and relatively prime to N?

6. A square is inscribed in a circle, and a rectangle is inscribed in the square. Another circle is circumscribed about the rectangle, and a smaller circle is tangent to three sides of the rectangle, as shown below. The shaded area between the two larger circles is eight times the area of the smallest circle, which is also shaded. What fraction of the largest circle is shaded?



7. In $\triangle ABC$, the length of AB is 12 and its incircle O has radius 4. Let D be the point of tangency of circle O with AB. If $AD : AB = 1 : 3$, find the area of $\triangle ABC$.

8. Suppose that $\{a_n\}_{n\geq 1}$ is an arithmetic sequence of real numbers such that

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_{10} = 20,$$
$$a_1 + a_4 + a_9 + a_{16} + \cdots + a_{100} = 18.$$

Compute $a_1 + a_8 + a_{27} + a_{64} + \cdots + a_{1000}$.

9. Let α and β be the roots of the equation $x^2 - 11x + 24 = 0$. Let f be the polynomial of least degree, with integer coefficients and leading coefficient 1, such that $\sqrt{\alpha} + \sqrt{\beta}$ and $\sqrt{\alpha\beta}$ are zeros of f. Find $f(1)$.

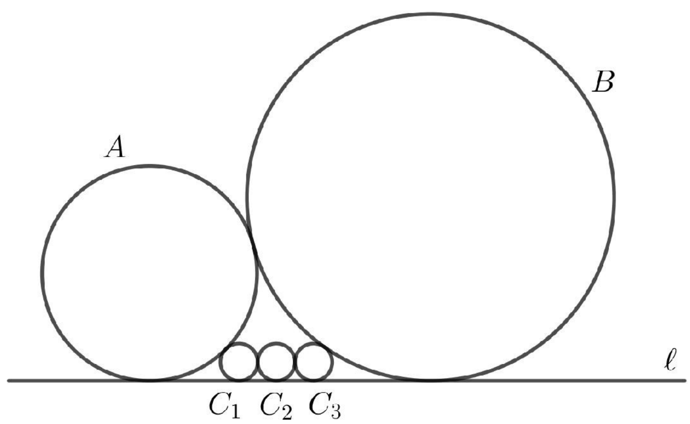
10. Suppose that the lengths of the sides of a right triangle are integers and its area is six times its perimeter. What is the least possible length of its hypotenuse?

11. A *Vitas word* is a string of letters that satisfies the following conditions:

    - It consists of only the letters B, L, R.
    - It begins with a B and ends in an L.
    - No two consecutive letters are the same.

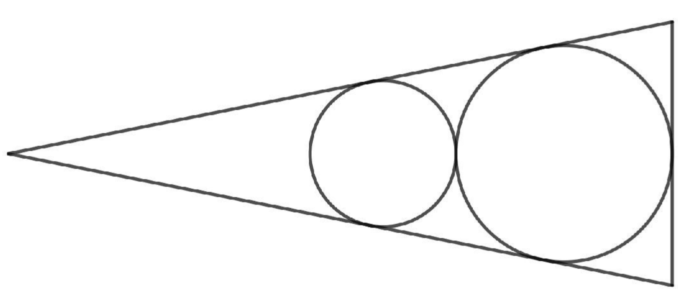
    How many Vitas words are there with 11 letters?

12. In the figure below, five circles are tangent to line ℓ. Each circle is externally tangent to two other circles. Suppose that circles A and B have radii 4 and 225, respectively, and that C_1, C_2, C_3 are congruent circles. Find their common radius.



13. Let $S = \{1, 2, 3, \ldots, 12\}$. Find the number of *nonempty* subsets T of S such that if $x \in T$ and $3x \in S$, then it follows that $3x \in T$.

14. In the figure below, the incircle of the isosceles triangle has radius 3. The smaller circle is tangent to the incircle and the two congruent sides of the triangle. If the smaller circle has radius 2, find the length of the base of the triangle.



15. Evaluate the expression $(1 + \tan 7.5^\circ)(1 + \tan 18^\circ)(1 + \tan 27^\circ)(1 + \tan 37.5^\circ)$.

16. Compute the number of ordered 6-tuples (a, b, c, d, e, f) of positive integers such that

$$a + b + c + 2(d + e + f) = 15.$$

17. Let $S = \{1, 2, \ldots, 2018\}$. For each subset T of S, take the product of all elements of T, with 1 being the product corresponding to the empty set. The sum of all such resulting products (with repetition) is N. Two elements m and n of S, with $m < n$, are removed. The sum of all products over all subsets of the resulting set is $\dfrac{N}{2018}$. What is n?

18. Let α be the unique positive root of the equation

$$x^{2018} - 11x - 24 = 0.$$

Find $\lfloor \alpha^{2018} \rfloor$. (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

19. How many distinct numbers are there in the sequence $\left\lfloor \dfrac{1^2}{2018} \right\rfloor, \left\lfloor \dfrac{2^2}{2018} \right\rfloor, \ldots, \left\lfloor \dfrac{2018^2}{2018} \right\rfloor$?

20. Suppose that a, b, c are real numbers such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 4\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) = \frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} = 4.$$

Determine the value of abc.

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. For a positive integer n, let $\varphi(n)$ denote the number of positive integers less than and relatively prime to n. Let $S_k = \sum_{n} \dfrac{\varphi(n)}{n}$, where n runs through all positive divisors of 42^k. Find the largest positive integer $k < 1000$ such that S_k is an integer.

2. In $\triangle ABC$, $AB > AC$ and the incenter is I. The incircle of the triangle is tangent to sides BC and AC at points D and E, respectively. Let P be the intersection of the lines AI and DE, and let M and N be the midpoints of sides BC and AB, respectively. Prove that M, N, and P are collinear.

3. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ satisfying, for all $n \in \mathbb{N}$,

   (a) $|f(n)| = n$

   (b) $0 \leq \sum_{k=1}^{n} f(k) < 2n$.

   Evaluate $\sum_{n=1}^{2018} f(n)$.

Answers to the 21st PMO Area Stage

Part I. (3 points each)

1. 7
2. 3
3. 5
4. 77220
5. 48384
6. $\frac{9}{25}$
7. 96
8. 2
9. -92
10. 58
11. 341
12. $\frac{9}{4}$
13. 1151
14. $3\sqrt{6}$
15. 4
16. 119
17. 1008
18. 35
19. 1514
20. $\frac{49}{23}$

Part II. (10 points each, full solutions required)

1. For a positive integer n, let $\varphi(n)$ denote the number of positive integers less than and relatively prime to n. Let $S_k = \sum_n \frac{\varphi(n)}{n}$, where n runs through all positive divisors of 42^k. Find the largest positive integer $k < 1000$ such that S_k is an integer.

   Answer: 996

   Solution: The function φ is the well-known Euler totient function which satisfies the property

$$\frac{\varphi(n)}{n} = \prod_{\substack{p \mid n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)$$

for any integer $n > 2$. Note that the problem defines $\varphi(1) = 0$.

For any $k \in \mathbb{N}$, the number 42^k has $(k+1)^3$ factors, each of which takes the form $2^a 3^b 7^c$ where $a, b, c \in \{0, 1, 2, \ldots, k\}$. Since $\varphi(n)/n$ depends only on the prime factors of n, we partition this set of factors into 8 forms with the same value for $\varphi(n)/n$.

	form of n	number of such ns	$\frac{\varphi(n)}{n}$	contribution to the sum
1	1	1	0	0
2	$2^a;\ a = 1, 2, \ldots, k$	k	$\left(1 - \frac{1}{2}\right) = \frac{1}{2}$	$\frac{k}{2}$
3	$3^b;\ b = 1, 2, \ldots, k$	k	$\left(1 - \frac{1}{3}\right) = \frac{2}{3}$	$\frac{2k}{3}$
4	$7^c;\ c = 1, 2, \ldots, k$	k	$\left(1 - \frac{1}{7}\right) = \frac{6}{7}$	$\frac{6k}{7}$
5	$2^a 3^b;\ a, b = 1, 2, \ldots, k$	k^2	$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{1}{3}$	$\frac{k^2}{3}$
6	$2^a 7^c;\ a, c = 1, 2, \ldots, k$	k^2	$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{7}\right) = \frac{3}{7}$	$\frac{3k^2}{7}$
7	$3^b 7^c;\ b, c = 1, 2, \ldots, k$	k^2	$\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right) = \frac{4}{7}$	$\frac{4k^2}{7}$
8	$2^a 3^b 7^c,\ a, b, c = 1, 2, \ldots, k$	k^3	$\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right) = \frac{2}{7}$	$\frac{2k^3}{7}$

Therefore, $S_k = \frac{k}{2} + \frac{2k}{3} + \frac{6k}{7} + \frac{k^2}{3} + \frac{3k^2}{7} + \frac{4k^2}{7} + \frac{2k^3}{7} = \frac{a(k)}{42}$, where $a(k) = 85k + 56k^2 + 12k^3$. Hence, the problem wants us to find the largest $k < 10^3$ so that $a(k) \equiv 0 \pmod{42}$, or equivalently, $a(k) \equiv 0 \pmod{2}$, $a(k) \equiv 0 \pmod{3}$, and $a(k) \equiv 0 \pmod{7}$. Observe that

- $a(k) \equiv k \pmod{2}$, which is 0 iff k is even.
- $a(k) \equiv k + 2k^2 \pmod{3}$, which is 0 iff $k \equiv 0$ or $1 \pmod{3}$
- $a(k) \equiv k + 5k^3 \pmod{7}$, which is 0 iff $k \equiv 0, 2,$ or $5 \pmod{7}$.

The numbers 999 and 997 are not even. $998 \equiv 2 \pmod{3}$. 996 is even, $\equiv 0 \pmod{3}$, and $\equiv 2 \pmod{7}$. Therefore, the answer is 996.

2. In $\triangle ABC$, $AB > AC$ and the incenter is I. The incircle of the triangle is tangent to sides BC and AC at points D and E, respectively. Let P be the intersection of the lines AI and DE, and let M and N be the midpoints of sides BC and AB, respectively. Prove that M, N, and P are collinear.

Solution: Let $\alpha = \angle A$, $\beta = \angle B$, and $\gamma = \angle C$. We will show that $\angle BNP = \angle BNM$.

Claim: Points B, I, D, and P are concyclic.

Proof of Claim:

Since $\triangle DCE$ is isosceles with $CD = CE$,

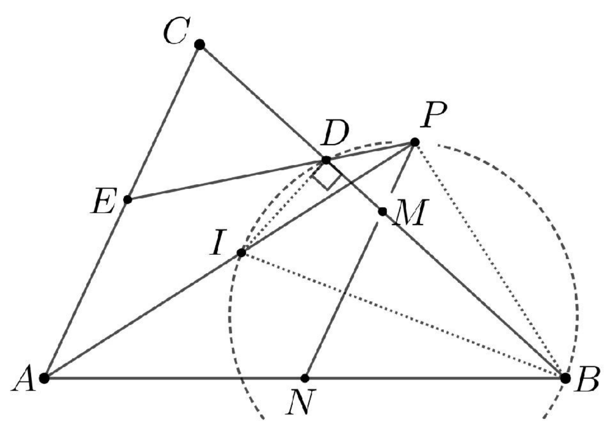
$$\angle BDP = \angle CDE = \frac{1}{2}(180^\circ - \gamma) = \frac{1}{2}(\alpha + \beta).$$

On the other hand,

$$\angle BIP = \angle BAI + \angle IBA = \frac{\alpha}{2} + \frac{\beta}{2}.$$

Therefore, $\angle BDP = \angle BIP$. Thus, B, I, D, and P are concyclic.

Alternative: $\angle DPI = \angle EPA = \angle CED - \angle EAP = \frac{1}{2}(180^\circ - \gamma) - \frac{1}{2}\alpha = \frac{1}{2}\beta = \angle DBI$. Thus, B, I, D, and P are concyclic. ■



This implies that since $ID \perp BC$, then $\angle APB = \angle IPB = \angle IDB = 90^\circ$. In right $\triangle APB$, N is the midpoint of the hypotenuse so it follows that $NP = NA$.

Consequently, $\angle BNP = \angle NAP + \angle APN = 2\angle NAP = \alpha$. Alternatively, since $\angle APB = 90^\circ$, then AB is a diameter of the circumcircle of $\triangle APB$, and N is the circumcenter. Consequently, $\angle BNP = 2\angle BAP = \alpha$.

Since M and N are the midpoints of AB and CB respectively, $\angle BNM = \angle BAC = \alpha$.

We now have $\angle BNP = \angle BNM$. Therefore, P, M, and N are collinear.

Alternative Approaches via the introduction of a phantom point

- Extend AI and NM to meet at P'. Goal: Show $P = P'$ by showing that P' is on line ED.
- Extend ED and NM to meet at P''. Goal: Show $P = P''$ by showing that P'' is on line AI.

3. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ satisfying, for all $n \in \mathbb{N}$,

   (a) $|f(n)| = n$

   (b) $0 \leq \sum_{k=1}^{n} f(k) < 2n$.

   Evaluate $\sum_{n=1}^{2018} f(n)$.

   Answer: 2649

   Solution: Let $S_n = \sum_{k=1}^{n} f(k)$. We want the value of S_{2018}.

   Claim: $f(n) = \begin{cases} n & \text{if } S_{n-1} < n \\ -n & \text{if } S_{n-1} \geq n \end{cases}$

   Proof: The inequality condition is $0 \leq S_{n-1} + f(n) < 2n$.

   - If $n > S_{n-1}$, then $0 \leq S_{n-1} + f(n) < n + f(n)$ so $f(n) > -n$. Therefore, $f(n) = n$.
   - If $n \leq S_{n-1}$, then $n + f(n) \leq S_{n-1} + f(n) < 2n$ so $f(n) < n$. Therefore, $f(n) = -n$.

   Claim: If $S_n = 0$, then

S_{n+1}	S_{n+2}	S_{n+3}	S_{n+4}	$\cdots$	S_{n+2j}	S_{n+2j+1}	$\cdots$
$n+1$	$2n+3$	n	$2n+4$	$\cdots$	$2n+2+j$	$n+1-j$	$\cdots$

   where $j = 1, 2, \ldots, n+1$.

   The pattern here: $S_{n+1}, S_{n+3}, S_{n+5}, \ldots$ are numbers decreasing by 1, while $S_{n+2}, S_{n+4}, S_{n+6}, \ldots$ are numbers increasing by 1.

   Proof: $S_n = 0 < n$ so $f(n+1) = n+1$. Thus, $S_{n+1} = 0 + (n+1) = n+1$.

   $S_{n+1} = n+1 < n+2$ so $f(n+2) = n+2$. Thus, $S_{n+2} = n+1+(n+2) = 2n+3$.

   $S_{n+2} = 2n+3 > n+3$ so $f(n+3) = -n-3$. Thus, $S_{n+3} = 2n+3+(-n-3) = n$.

   We prove the claim by strong induction. Suppose the pattern holds for $S_{n+1}, S_{n+2}, \ldots, S_{n-1+2j}$.

   Since $S_{n-1+2j} = S_{n+1+2(j-1)} = n+1-(j-1) = n+2-j < n+2j$, then $f(n+2j) = n+2j$ so $S_{n+2j} = (n+2-j)+(n+2j) = 2n+2+j$.

   On the other hand, since $S_{n+2j} = 2n+2+j = (n+2j+1)+(n+1-j) \geq n+2j+1$, then $f(n+2j+1) = -(n+2j+1)$ so $S_{n+2j+1} = (2n+2+j)-(n+2j+1) = n+1-j$, which proves the claim.

   Eventually, $S_{n+1}, S_{n+3}, \ldots, S_{n+2j+1}, \ldots$ will decrease to 0, when $j = n+1$. Thus, if $S_n = 0$, it follows that the next 0 value is $S_{3(n+1)}$.

   Therefore, $S_3 = 0$, $S_{3\cdot 4} = S_{12} = 0$, $S_{3\cdot 13} = S_{39} = 0$, $S_{3\cdot 40} = S_{120} = 0$, $S_{3\cdot 121} = S_{363} = 0$, $S_{3\cdot 364} = S_{1092} = 0$.

   Since $2018 = 1092 + 2 \cdot 463$, then $S_{2018} = 2 \cdot 1092 + 2 + 463 = 2649$.

HARI
HYUNDAI ASIA RESOURCES, INC.
FOUNDATION

HARI Foundation, Inc. Celebrates the Power of the Heart to Move Lives

From its humble beginnings over a decade ago, Hyundai Asia Resources, Inc. (HARI), the official Philippine distributor of Hyundai vehicles, has earned its place as a leader in the automotive industry. More than just reaping success as a business, HARI has taken upon itself to be a catalyst of change in Philippine society, inspired by the global vision of Hyundai Motor Company (HMC) to build a better world for all.

Through its Corporate Social Responsibility (CSR) arm, HARI Foundation, Inc. (HFI), HARI has engaged employees, dealerships, and customers, as well as partners in the public and private sectors in trail-blazing endeavors across the country in the areas of education, environment, community development, healthcare, and women's and children's rights.

Among HFI's roster of partners in nation building are Gawad Kalinga, St. Scholastica's Priory, the Department of Science and Technology (DOST), National Museum, The Mind Museum, Synergeia Foundation, the ASEAN Center for Biodiversity, and UP Philippine General Hospital.

Fully aware that poverty is more than just the lack of resources, but of options to pursue a better life, HFI has streamlined its efforts to a more targeted, long-term, multi-disciplinary, and multi-stakeholder solution. Education is at the core of HFI's contribution to the Philippine agenda for sustainability.

Says HFI President Ma. Fe Perez-Agudo, "I have always been a believer in empowering people through education. By engaging with like-minded partners from the government and the private sectors, we aim to be the Filipino's lifetime partner in building a sustainable, empowered nation one community at a time."

HFI's over 10 years of working with and learning from its various CSR partners and beneficiaries have led to its developing programs that respond to the needs of our times.

The Hyundai New Thinkers Circuit (HNTC), designed in partnership with the DOST, served as an innovative science literacy program that provokes, fosters, and nurtures leadership and the innovative spirit among outstanding high school students who can take the lead in building a climate change-resilient Philippines. Launched in 2013, HNTC yielded 11 scholars who are pursuing studies in the sciences at the country's top universities.

On March 22, 2017, HFI, Hyundai Motor Company (HMC) Korea, the Institute for Global Education, Exchange and Internship (IGEEI), and the Tanay local government collaborated to build the pilot Rain Water Harvesting System in Rawang Elementary School in Tanay City. This filtration method, an invention of Prof. Han Moo Young of Seoul National University, is capable of producing and storing potable water from rain gathered in roof gutters. Some 200 students of Rawang Elementary School were among its first beneficiaries.This project, which will soon be replicated in other under-served communities in the country, bagged for HFI the 2017 Gold Award from the Society of Philippine Motoring Journalists (SPMJ).

HFI has likewise been an active supporter of Gawad Kalinga (GK) efforts at rehabilitating war-torn Marawi through the donation of vehicles that ply the province as Kusina ng Kalinga (KNK) vans. KNK is GK's platform to address malnutrition among children in public schools, on the streets, and in conflict areas. HFI is taking a step further to help Marawi back to its feet with the rebuilding of public schools in strategic areas of the province.

Finally, HFI in partnership with UP-PGH and the UP-PGH Cancer Institute, aims to empower women through proactive healthcare. Hyundai in the Philippines, through HFI, has donated a Hyundai H350 luxury van customized into a state-of-the-art mobile cancer diagnostic clinic that will transport UP-PGH medical missions to under-served sectors of the country. Dubbed the Alagang Breastfriend project, this comprehensive breast cancer awareness campaign is envisioned to provide women information and access to important resources and technology that would improve their overall well-being, thereby enabling them to lead healthier, more productive lives.

"In a nutshell, the story of HFI in this new century goes by the acronym H.E.A.R.T.—Health, Education, Arts, Rebuilding, and Transformative Leadership. We kick off a new leg in our journey to broaden our reach and design programs that are more meaningful to people, especially at the grassroots", concluded Ms. Agudo.

More than a foundation, HFI represents a movement to build a better world for all, one community at a time, driven by faith in the power of the Filipino heart to innovate, engage, and collaborate to drive shared dreams forward and give rise to generations of leaders capable of advancing sustainability in all important aspects of our lives.

Manulife

Manulife Business Processing Services is the global shared service center of Manulife which provides actuarial, analytics, data science, administrative, finance, investments, marketing, underwriting, and information technology services to Manulife and John Hancock operations in Canada, the United States, and Asia. MBPS has been in the Philippines for 12 years with sites in Quezon City and Cebu.

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F·U·S·E

Founded by Dr. Lucio C. Tan on December 1, 1994, FUSE seeks to improve the skills in English, science and Mathematics teachers. Its programs include regular teacher training workshops; a post-graduate scholarship program for Science and Engineering teachers; production and distribution of telecourses in English, Elementary Science, Chemistry, Physics and Math; and a host of other educational programs in partnership with government and private learning institutions.


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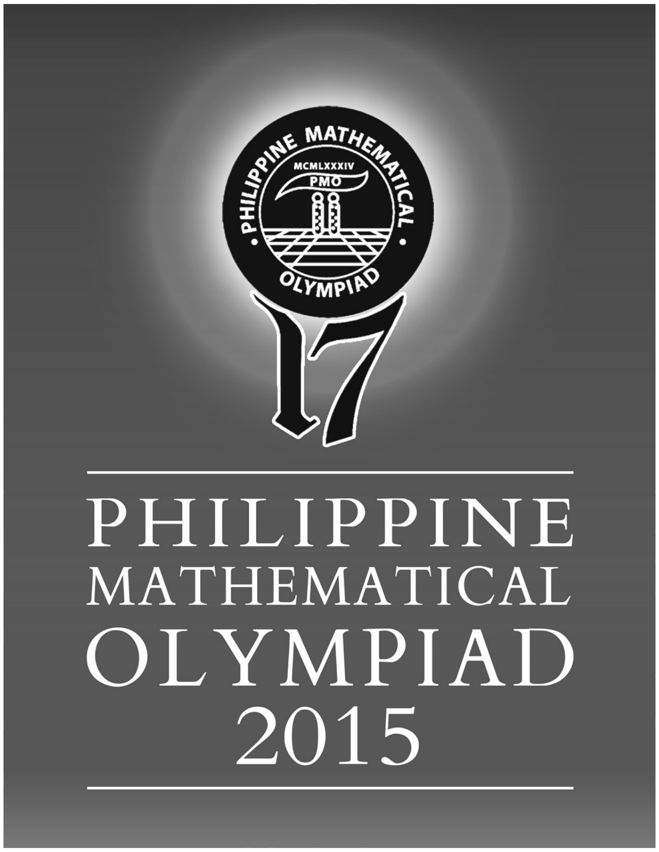
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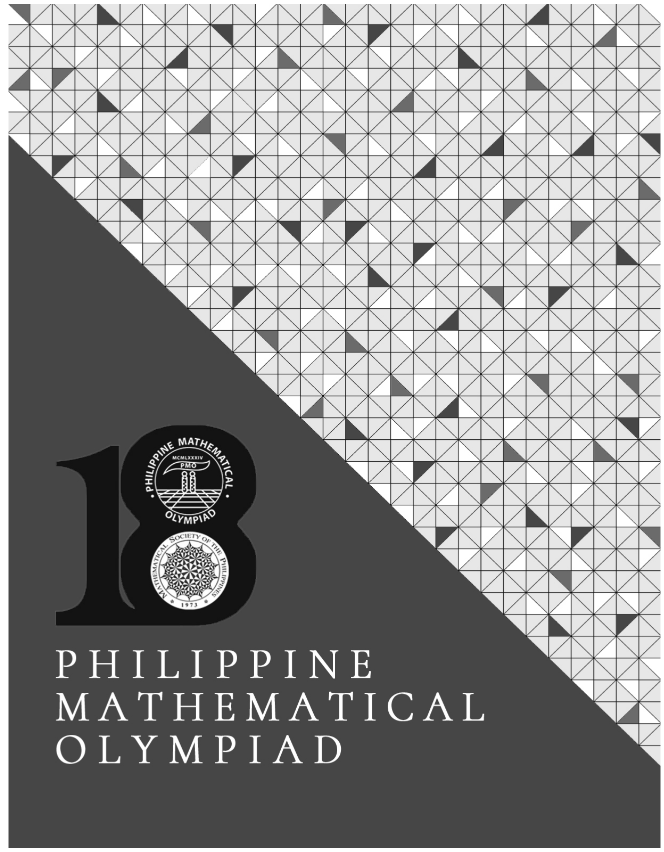
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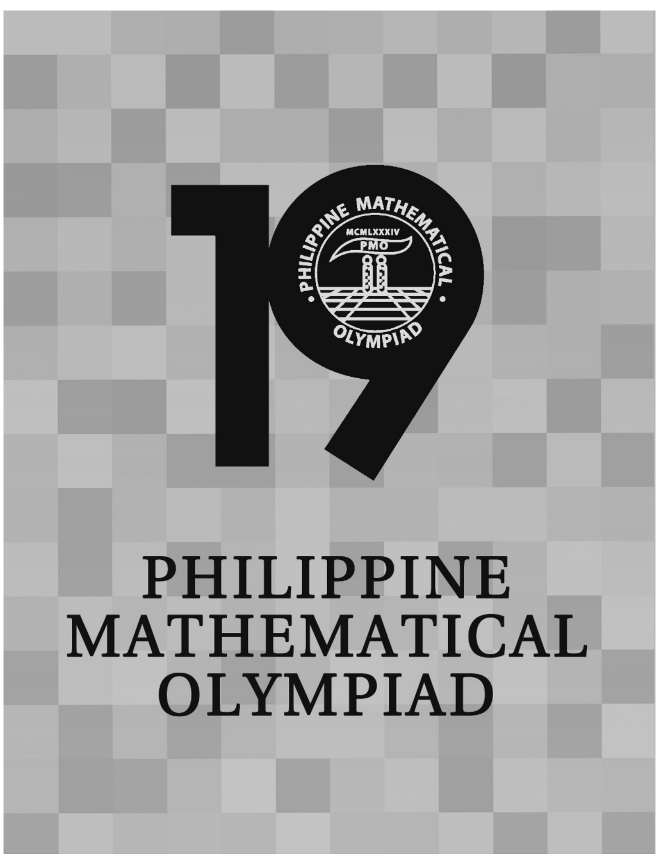
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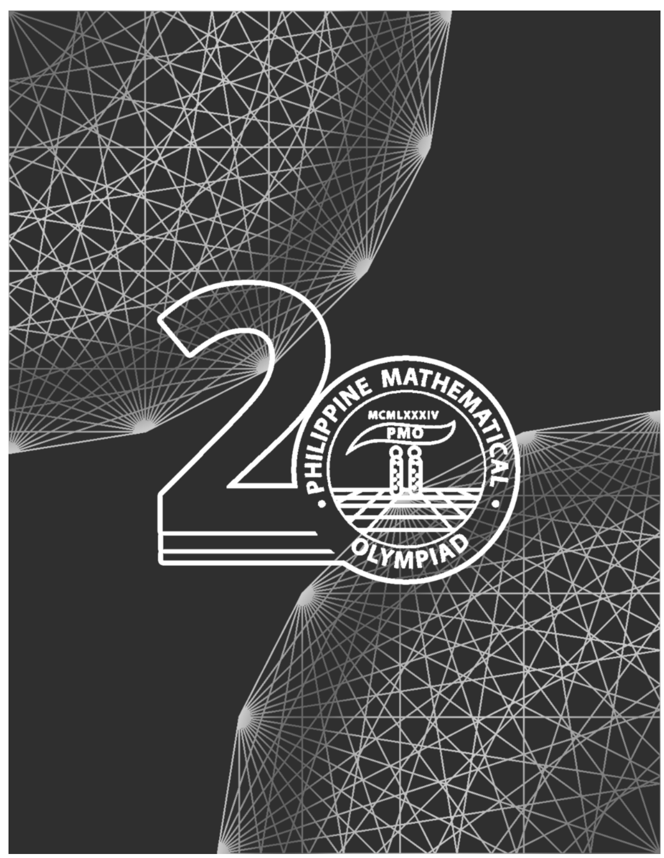
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