19th Philippine Mathematical Olympiad

Area Stage

19 November 2016

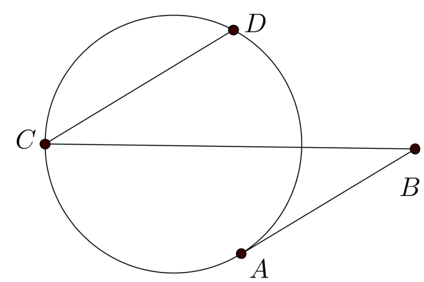
PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. The vertices of a triangle are at the points $(0, 0)$, (a, b), and $(2016 - 2a, 0)$, where $a > 0$. If (a, b) is on the line $y = 4x$, find the value(s) of a that maximizes the triangle's area.

2. Let f be a real-valued function such that

$$f(x - f(y)) = f(x) - xf(y)$$

for any real numbers x and y. If $f(0) = 3$, determine $f(2016) - f(2013)$.



3. In the figure on the right, AB is tangent to the circle at point A, BC passes through the center of the circle, and CD is a chord of the circle that is parallel to AB. If $AB = 6$ and $BC = 12$, what is the length of CD?

4. Suppose that S_k is the sum of the first k terms of an arithmetic sequence with common difference 3. If the value of $\dfrac{S_{3n}}{S_n}$ does not depend on n, what is the 100th term of the sequence?

5. In parallelogram $ABCD$, $AB = 1$, $BC = 4$, and $\angle ABC = 60°$. Suppose that AC is extended from A to a point E beyond C so that triangle ADE has the same area as the parallelogram. Find the length of DE.

6. Find the exact value of $\tan^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}\left(\dfrac{1}{5}\right) + \tan^{-1}\left(\dfrac{1}{8}\right)$.

7. A small class of nine boys are to change their seating arrangement by drawing their new seat numbers from a box. After the seat change, what is the probability that there is only one pair of boys who have switched seats with each other and only three boys who have unchanged seats?

8. For each $x \in \mathbb{R}$, let $\{x\}$ be the fractional part of x in its decimal representation. For instance, $\{3.4\} = 3.4 - 3 = 0.4$, $\{2\} = 0$, and $\{-2.7\} = -2.7 - (-3) = 0.3$. Find the sum of all real numbers x for which $\{x\} = \dfrac{1}{5}x$.

9. Find the integer which is closest to the value of $\dfrac{1}{\sqrt[6]{5^6+1}-\sqrt[6]{5^6-1}}$.

10. A line intersects the y-axis, the line $y = 2x + 2$, and the x-axis at the points A, B, and C, respectively. If segment AC has a length of $4\sqrt{2}$ units and B lies in the first quadrant and is the midpoint of segment AC, find the equation of the line in slope-intercept form.

11. How many real numbers x satisfy the equation

$$\left(|x^2 - 12x + 20|^{\log x^2}\right)^{-1+\log x} = |x^2 - 12x + 20|^{1+\log(1/x)}?$$

12. Let $n = 2^{23}3^{17}$. How many factors of n^2 are less than n, but do not divide n?

13. A circle is inscribed in a 2 by 2 square. Four squares are placed on the corners (the spaces between circle and square), in such a way that one side of the square is tangent to the circle, and two of the vertices lie on the sides of the larger square. Find the total area of the four smaller squares.

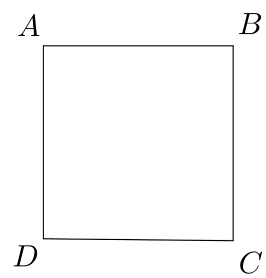
14. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (2x - y, x + 2y)$. Let $f^0(x, y) = (x, y)$ and, for each $n \in \mathbb{N}$, $f^n(x, y) = f(f^{n-1}(x, y))$. Determine the distance between $f^{2016}\left(\dfrac{4}{5}, \dfrac{3}{5}\right)$ and the origin.

15. How many numbers between 1 and 2016 are divisible by exactly one of 4, 6, or 10?

16. Let N be a natural number whose base-2016 representation is ABC. Working now in base-10, what is the remainder when $N - (A + B + C + k)$ is divided by 2015, if $k \in \{1, 2, \ldots, 2015\}$?

17. Find the number of pairs of positive integers (n, k) that satisfy the equation $(n+1)^k - 1 = n!$.

18. A railway passes through four towns A, B, C, and D. The railway forms a complete loop, as shown on the right, and trains go in both directions. Suppose that a trip between two adjacent towns costs one ticket. Using exactly eight tickets, how many distinct ways are there of traveling from town A and ending at town A? (*Note that passing through A somewhere in the middle of the trip is allowed.*)



19. The lengths of the two legs of a right triangle are in the ratio of 7 : 24. The distance between its incenter and its circumcenter is 1. Find its area. (*Recall that the* ***incenter*** *of a triangle is the center of its inscribed circle and the* ***circumcenter*** *is the center of its circumscribing circle.*)

20. Let $\lfloor x \rfloor$ be the greatest integer not exceeding x. For instance, $\lfloor 3.4 \rfloor = 3$, $\lfloor 2 \rfloor = 2$, and $\lfloor -2.7 \rfloor = -3$. Determine the value of the constant $\lambda > 0$ so that $2\lfloor \lambda n \rfloor = 1 - n + \left\lfloor \lambda \lfloor \lambda n \rfloor \right\rfloor$ for all positive integers n.

PART II. Show your solution to each problem. Each complete and correct answer is worth ten points.

1. Let x and y be real numbers that satisfy the following system of equations:

$$\begin{cases} \dfrac{x}{x^2y^2-1} - \dfrac{1}{x} &= 4 \\ \dfrac{x^2y}{x^2y^2-1} + y &= 2 \end{cases}$$

   Find all possible values of the product xy.

2. Let BH be the altitude from the vertex B to the side AC of an acute-angled triangle ABC. Let D and E be the midpoints of AB and AC, respectively, and F the reflection of H across the line segment ED. Prove that the line BF passes through the circumcenter of ΔABC.

3. A function $g : \mathbb{N} \to \mathbb{N}$ satisfies the following:

   (a) If m is a proper divisor of n, then $g(m) < g(n)$.

   (b) If m and n are relatively prime and greater than 1, then

$$g(mn) = g(m)g(n) + (n+1)g(m) + (m+1)g(n) + m + n.$$

   Find the least possible value of $g(2016)$.

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PART I. (3 points each)

1. 504
2. 6048
3. 7.2 units
4. 597/2 or 298.5
5. $2\sqrt{3}$
6. $\dfrac{\pi}{4}$
7. $\dfrac{1}{48}$
8. $\dfrac{15}{2}$
9. 9375
10. $y = -7x + \dfrac{28}{5}$
11. 6
12. 391
13. $\dfrac{48 - 32\sqrt{2}}{9}$
14. 5^{1008}
15. 470
16. $2015 - k$
17. 3
18. 128
19. $\dfrac{336}{325}$
20. $1 + \sqrt{2}$

PART II. (10 points each)

1. The given system can be expressed as follows:

$$\begin{cases} \dfrac{x}{x^2y^2-1} - \frac{1}{x} & = & 4 \\ \dfrac{x^2y}{x^2y^2-1} + y & = & 2 \end{cases} \Rightarrow \begin{cases} \dfrac{x^2}{x^2y^2-1} - 1 & = & 4x \\ \dfrac{x^2}{x^2y^2-1} + 1 & = & \dfrac{2}{y} \end{cases}$$

We then have

$$4x + \frac{2}{y} = \frac{2x^2}{x^2y^2-1} \Rightarrow 2x + \frac{1}{y} = \frac{x^2}{x^2y^2-1}$$

and

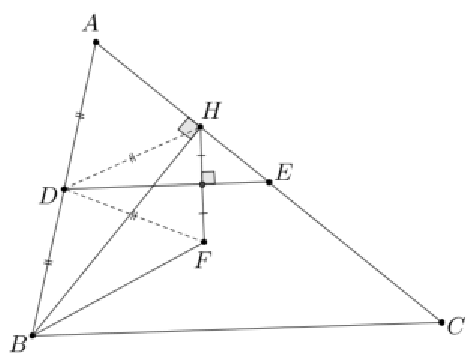
$$4x - \frac{2}{y} = -2 \Rightarrow 2x - \frac{1}{y} = -1,$$

which gives us

$$
\begin{aligned}
\left(2x+\frac{1}{y}\right)\left(2x-\frac{1}{y}\right) &= \frac{-x^2}{x^2y^2-1}\\
4x^2-\frac{1}{y^2} &= \frac{-x^2}{x^2y^2-1}\\
\frac{4x^2y^2-1}{y^2} &= \frac{-x^2}{x^2y^2-1}\\
\left(4x^2y^2-1\right)\left(x^2y^2-1\right) &= -x^2y^2\\
4x^4y^4-5x^2y^2+1 &= -x^2y^2\\
4x^4y^4-4x^2y^2+1 &= 0\\
\left(2x^2y^2-1\right)^2 &= 0 \Rightarrow x^2y^2=\frac{1}{2} \Rightarrow \boxed{xy=\pm\frac{1}{\sqrt{2}}}
\end{aligned}
$$

2. This problem is taken from the ***2015 Iranian Geometry Olympiad***.

   Solution 1. Let O be the circumcenter of ΔABC. Since $\angle OBA = 90^\circ - \angle C$, it suffices to show that $\angle FBA = 90^\circ - \angle C$.

   Note that $AD = BD = DH$ and $DH = DF$. Therefore, quadrilateral $AHFB$ is cyclic (with circumcenter D), and so $\angle FBA = \angle FHE = 90^\circ - \angle DEH$. Since DE is parallel to BC, $\angle DEH = \angle C$, and $\angle FBA = 90^\circ - \angle C$. □

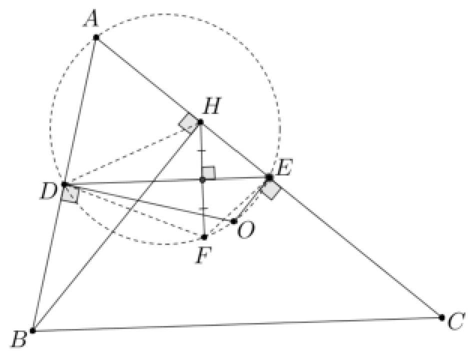
   Solution 2. As before, denote by O the circumcenter of ΔABC. Then the quadrilateral $ADOE$ is cyclic. Also, we know that $AD = HD = DB$, therefore,

$$\angle A = \angle DHA = 180^\circ - \angle DHE = 180^\circ - \angle DFE$$

   Therefore, $ADFE$ is cyclic. Since $ADFOE$ is cyclic, $DFOE$ is also cyclic, and

$$\angle C = \angle DEA = \angle DEF = \angle DOF$$

   On the other hand, $\angle C = \angle DOB$, so $\angle DOF = \angle DOB$, therefore B, F, and O are collinear. □



3. Consider $h(x) := g(x) + x + 1$. We have that, for m, n coprime and greater than 1,

$$\begin{aligned} h(m)h(n) &= (g(m) + m + 1)(g(n) + n + 1) \\ &= g(m)g(n) + (n+1)g(m) + (m+1)g(n) + mn + m + n + 1 \\ &= g(mn) + mn + 1 \\ &= h(mn). \end{aligned}$$

Repeating this, we find that more generally, if $m_1, m_2, \dots, m_k$ are pairwise coprime positive integers all greater than 1,

$$h\left(\prod_{i=1}^{k} m_i\right) = \prod_{i=1}^{k} m_i.$$

Hence, it suffices to consider h, and thus g, only on prime powers. Since

$$g(p^n) > g(p^{n-1}) > \cdots > g(p) > g(1) \geq 1,$$

we have $g(p^n) \geq n + 1$. Indeed, taking $g(1) = 1$, $g(p^n) = n + 1$ gives us a well-defined function g on $\mathbb{N}$. To solve for $g(2016)$, we solve for $h(2016)$ first, noting that $2016 = 2^5 \cdot 3^2 \cdot 7^1$:

$$\begin{aligned} h(2016) &= h(2^5)h(3^2)h(7^1) \\ &= (7 + 2^5)(4 + 3^2)(3 + 7^1) \\ &= 5070 \end{aligned}$$

and so $g(2016) = 5070 - 2017 = \boxed{3053}$. This is the minimum possible value of $g(2016)$. $\square$