# PHILIPPINE MATHEMATICAL OLYMPIAD

# PHILIPPINE MATHEMATICAL OLYMPIAD 2016

***NATIONAL STAGE***

***23 January 2016***

***University of Santo Tomas, Manila***

# SCHEDULE

| | |
|---|---|
| 7:30 AM - 8:30 AM | ***Registration*** |
| 9:00 AM - 12:00 NN | ***Phase I : Written Phase*** |
| 12:00 NN - 1:00 PM | ***Lunch Break*** |
| 1:00 PM - 5:00 PM | ***Phase II : Oral Phase*** |
| | ***National Anthem*** |
| | ***Welcoming Remarks*** |
| | ***Oral Competition*** |
| 6:00 PM - 8:30 PM | ***Dinner and Awarding Ceremonies*** |

# ABOUT THE PMO

First held in ***1984***, the PMO was created as a venue for high school students with interest and talent in mathematics to come together in the spirit of friendly competition and sportsmanship. Its aims are: ***(1)*** to awaken greater interest in and promote the appreciation of mathematics among students and teachers; ***(2)*** to identify mathematically-gifted students and motivate them towards the development of their mathematical skills; ***(3)*** to provide a vehicle for the professional growth of teachers; and ***(4)*** to encourage the involvement of both public and private sectors in the promotion and development of mathematics education in the Philippines.

The PMO is the first part of the selection process leading to participation in the ***International Mathematical Olympiad (IMO)***. It is followed by the ***Mathematical Olympiad Summer Camp (MOSC)***, a five-phase program for the twenty national finalists of PMO. The four selection tests given during the second phase of MOSC determine the tentative Philippine Team to the IMO. The final team is determined after the third phase of MOSC.

The PMO this year is the seventeenth since 1984. Four thousand six hundred twenty two (4622) high school students from all over the country took the qualifying examination, out of these, two hundred twenty four (224) students made it to the Area Stage. Now, in the National Stage, the number is down to twenty and these twenty students will compete for the top three positions and hopefully move on to represent the country in the ***57th IMO***, which will be held in ***Hong Kong***, from ***6 to 16 July 2015*** .

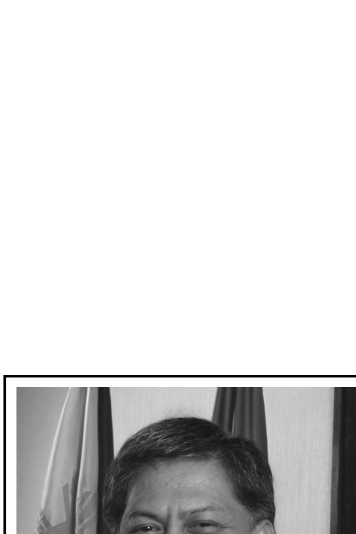
# MESSAGE

We welcome the participants to the 18th Philippine Mathematical Olympiad, hosted by the Mathematical Society of the Philippines (MSP). The Department of Education (DepEd) lauds the MSP for its continued commitment in encouraging young people to recognize the many applications of mathematics in our daily lives. Indeed, we cannot discount the importance of mathematics as we apply its principles and concepts in almost everything that we do.

It is also an area of study that promotes a better understanding of the physical world we live in even as it helps hone our critical thinking.

We are therefore grateful to the men and women behind this event for putting together an annual activity that will further develop greater appreciation for mathematics being one of the critical building blocks in creating an economically and technically advanced society.

*Patuloy tayong magsama-sama para sa Edukasyon!*



***BR. ARMIN A. LUISTRO, FSC***

*Secretary*

*Department of Education*

# MESSAGE

Greetings of peace!

We are glad to welcome 2016 with the highly anticipated Final Round of the country's most prestigious mathematics competition, the Philippine Mathematical Olympiad (PMO).

The nation's oldest competition, now in its 18th year, has continuously proven its value not only in discovering and advancing exemplary talents in mathematics, but also in training our would-be medalists in the grandest stage, the International Mathematical Olympiad (IMO). For almost a decade now, the PMO has enabled the country to carry on its medal haul in the IMO, the most prestigious mathematics competition in the world. Each year, we perform better and we are getting closer to winning our first gold medal. But more than that, it is the improved quality and competitiveness in mathematics shown by our students that stands out as the best impact of the PMO. Indeed, the high degree of competition in the PMO brings out the paramount potential of our students in mathematics.

As medals from international math and science competitions keep on pouring for the Philippines, expectations soar even higher. We believe that the Filipino students have the adequate knowledge and analytic skills that are at par with their international counterparts. And with the full realization of the K-12 program, the Philippine contingent will soon rise above its contenders and finally dominate in the international scene.

The Science Education Institute upholds its commitment in supporting the Mathematics Society of the Philippines in nurturing the Filipino youth who are gifted in mathematics. We are positive that this year's PMO will harvest a new crop of outstanding changemakers in the S&T landscape.

We look forward to an exciting PMO and we wish all the contestants the best.

***JOSETTE T. BIYO, PH. D.***

*Director*

*Science Education Institute, DOST*

# MESSAGE

Greetings to all our brilliant students, dedicated coaches and supportive school administrators to the 18th National Finals of the Philippine Mathematical Olympiad. The Mathematical Society of the Philippines is proud to once again conduct this endearing and prestigious annual tradition that brings together some of the country's most talented students in a friendly competition. I hope that this year's PMO has been fun and memorable for everyone.

Having bested hundreds of talented students in the regional and area stages, you have earned a rightful place in this most coveted final round. Congratulations to all of you. Thank you too to your loving parents and teachers who have provided the environment for you to maximize your God-given talents.

The mathematician Siméon Poisson proclaimed that *"Life is good for only two things, discovering mathematics and teaching mathematics*." A rather extreme view, but for a good number of us in the MSP, Poisson's words give relief and a reason to smile in the midst of a stressful day. I hope you will always experience the thrills of discovering mathematics, and one day, follow in the footsteps of your teachers.

To the DOST-Science Education Institute, DepEd, FUSE, Sharp Calculators, C & E Publishing, and other sponsors and partners, our sincere gratitude for your continuing support and generosity.

*Mabuhay tayong lahat!*

***JOSE MARIA P. BALMACEDA, PH. D.***

*President*

*Mathematical Society of the Philippines*

# MESSAGE

Congratulations to the Philippine Mathematical Olympiad for holding yet another one of its prosperous and motivational competitions through these 17 years of passing on mathematical excellence to the future leaders of their generation!

I give my regards to the organizers and trainers who have devoted their 100% effort, determination, and time in all its planning and preparation in order to make this competition a success. Moreover, I commend the parents who continuously support their children and provide their needs in this challenging opportunity.



Most importantly, I would like to salute all these young and gifted mathematicians who persevered from the beginning of the competition until its final phase. May you strive to become the best you can be and soon represent the Philippines with your heads high!

*"And whatever you do in word or deed, do all in the name of the Lord Jesus, giving thanks to God the Father through Him."* -Colossians 3:17

***LUCERO ONG***

*Assistant Vice President*

*Sharp Calculators*

*Collins International Trading*

# MESSAGE

I salute the Mathematical Society of the Philippines! Because of its purposiveness and constancy in inspiring students to solve and learn mathematics as to excel even in international competitions, the Philippine Team's six (6) students ranked 36th among 104 participating countries, the Philippines' best since 1989. I share your serious intent to rise to the top.

The Mathematical Society of the Philippines has demonstrated that excelling in competitions is attainable. I specially wish it well in its continuing and fervent campaign to develop effective teachers of mathematics who will produce, among others, the winners of tomorrow.



Again, Congratulations!

***DR. LUCIO C. TAN***

*Vice-Chairman, Board of Trustees*

*Foundation for Upgrading the Standard of Education Inc.*

# MESSAGE

# The PMO Team

***DIRECTOR***
Richard Lemence

***ASSISTANT DIRECTORS***
Ma. Nerissa Masangkay Abara
Rigor Ponsones
Recto Rex Calingasan

***TREASURER***
Johnatan Pimentel

***NATIONAL SECRETARY***
May Anne Tirado

***SECRETARY***
Gaudella Ruiz

***LAYOUT ARTIST***
Rolando Perez III

***GENERAL DESIGNER***
Paul Reine Kennett Dela Rosa

***TEST DEVELOPMENT COMMITTEE***
Kristine Joy Carpio
Robert Neil Leong
Deneisha Gayle Tieng
Timothy Robin Teng
Christian Paul Chan Shio
John Gabriel Pelias
Mark Anthony Tolentino
Louie John Vallejo
Richard Eden

***NATIONAL STAGE PREPARATIONS***
Enrico M. Yambao
Josephine Bernadette M. Benjamin
Charles R. Repizo
Juliano S. Parena Jr.
Sheen Mclean S. Cabañeros
Kristan Bryan C. Simbulan
Rea Divina C. Mero
Mark Louie F. Ramos

***REGIONAL COORDINATORS***

***REGION I, CAR***
Jerico Bacani

***REGION II***
Crizaldy Binarao

***REGION III***
Jumar Valdez

***REGION IV-A***
Sharon Lubag

***REGION IV-B***
Shiela Grace Soriano

***REGION V***
Solomon Olayta

***REGION VI***
Salve Marie Fuentes

***REGION VII***
Lorna Almocera

***REGION VIII***
Ariel Salarda

***REGION IX***
Rochelleo Mariano

***REGION X, XII, ARMM***
Gina Malacas

***REGION XI***
Eveyth Deligero

***REGION XIII***
Miraluna Herrera

***NCR***
Lemuel Martin

# AREA STAGE WINNERS

***LUZON***

| | | |
|---|---|---|
| ***1*** | Albert John Patupat | Holy Rosary College |
| ***2*** | Vince Jan Torres | Santa Rosa Science and Technology High School |
| ***3*** | Errol John Suarez | Aquinas Univ. of Science and Technology High School |

***VISAYAS***

| | | |
|---|---|---|
| ***1*** | William Joshua King | Bethany Christian School |
| ***2*** | Myles Denzel Delatore | Bethany Christian School |
| ***2*** | Dominic Yap | Philippine Science High School - Western Visayas |
| ***3*** | Brigham Lucero | Philippine Science High School - Central Visayas |

***MINDANAO***

| | | |
|---|---|---|
| ***1*** | Xavier Jefferson Ray Go | Zamboanga Chong Hua High School |
| ***1*** | Sean Anderson Ty | Zamboanga Chong Hua High School |
| ***2*** | Vicente Raphael Chan | Zamboanga Chong Hua High School |
| ***3*** | Kenneth M. A. Antonio | Bayugan National Comprehensive High School |

***NCR***

| | | |
|---|---|---|
| ***1*** | Clyde Wesley Ang | Chiang Kai Shek College |
| ***2*** | Luke Matthews Bernardo | Philadelphia High School |
| ***2*** | Kyle Patrick Dulay | Philippine Science High School - Main |
| ***3*** | Carl Joshua Quines | Valenzuela City School of Science and Mathematics |

# ~ PRIZES ~

***The prizes for the TOP THREE for each Area/Region (Luzon, Visayas, Mindanao, NCR) are:***

***FIRST PLACE*** - Medal and SHARP EL-W531XH Calculator

***SECOND PLACE*** - Medal and SHARP EL-510RN Calculator

***THIRD PLACE*** - Medal and SHARP EL-501X Calculator

***The prizes for the TOP THREE in the NATIONAL FINAL STAGE are:***

***FIRST PLACE*** - P 20,000, Medal, and SHARP EL-738FB Calculator with SHARP Goodies

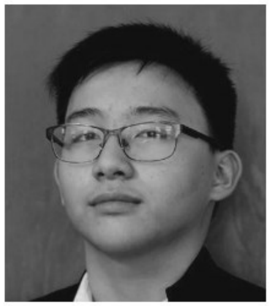
***SECOND PLACE*** - P 15,000, Medal, and SHARP EL-W506X Calculator with SHARP Goodies

***THIRD PLACE*** - P 10,000, Medal, and SHARP EL-W506X Calculator with SHARP Goodies

The coaches of the first, second and third placers will receive P 5,000, P 3,000, and P 2,000, respectively

# The PMO Finalists

***CLYDE WESLEY ANG***
Chiang Kai Shek College

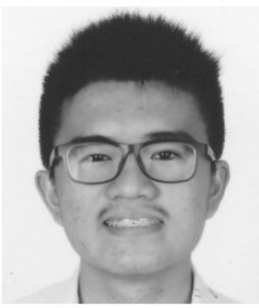


***LUKE MATTHEWS BERNARDO***
Philadelphia High School

***ELIJAMIN WOLFGANG CLAVERIA***
Philippine Science High School – Main

***KYLE PATRICK DULAY***
Philippine Science High School – Main

***CHRISTIAN PHILIP GELERA***
Philippine Science High School – Main



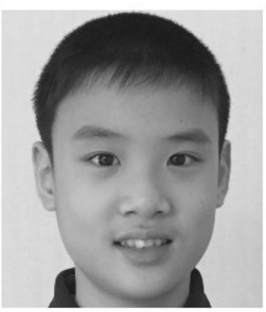
***XAVIER JEFFERSON RAY GO***
Zamboanga Chiong Hua High School

***ANDRES RICO GONZALES III***
Colegio de San Juan de Letran

***MATTHEW ANGELO ISIDRO***
Saint Jude Catholic School

***ANDREA JESSICA JABA***
Saint Jude Catholic School

***SEDRICK SCOTT KEH***
Xavier School



***DION STEPHAN ONG***
Ateneo de Manila High School

***TIFFANY MAE ONG***
Immaculate Concepcion Academy - Greenhills

***ALBERT JOHN PATUPAT***
Holy Rosary College

***SHAQUILLE WYAN QUE***
Grace Christian College

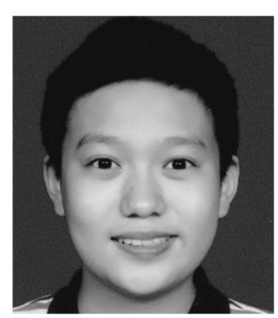
***CARL JOSHUA QUINES***
Valenzuela City School of Science and Mathematics

***ERROL JOHN SUAREZ***
Aquinas Univ. of Legaspi Sci. and Tech. High School

***VINCE JAN TORRES***
Santa Rosa Science and Technology High School

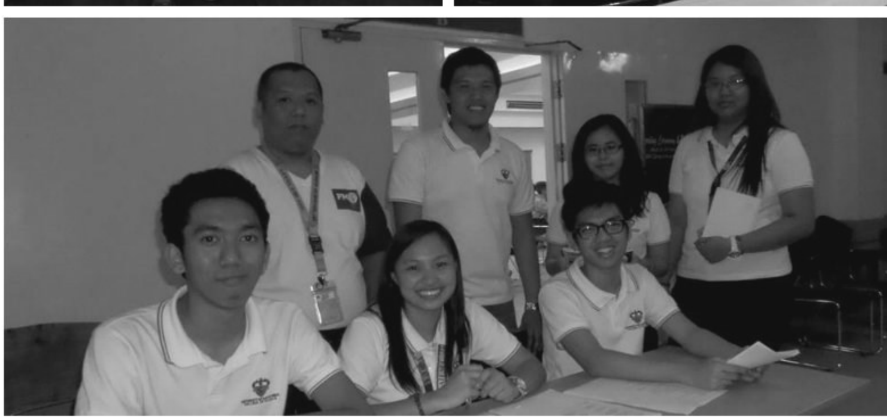
***SEAN ANDERSON TY***
Zamboanga Chiong Hua High School

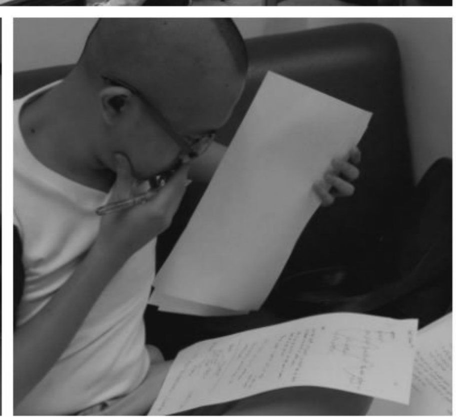
***ISABEL JOCYN VILLANUEVA***
PAREF Woodrose



***FARRELL ELDRIAN WU***
MGC New Life Christian Academy

# 17th PMO Highlights



# $18^{\text{th}}$ Philippine Mathematical Olympiad

*Qualifying Stage*

17 October 2015

**PART I.** Choose the best answer. Each correct answer is worth two points.

1. If $(x+5)(2x-3)=0$, what are all the possible values of $2x-3$?

   (a) $\frac{13}{2}$ only (b) 0 and $-13$ (c) 0 and $-5$ (d) $-5$ and $\frac{3}{2}$

2. What is the sum of all the even positive divisors of 64?

   (a) 61 (b) 62 (c) 126 (d) 128

3. The measures of the angles of a quadrilateral are $x$, $x+10$, $x+30$, and $x+40$ degrees. What is the largest angle, in degrees?

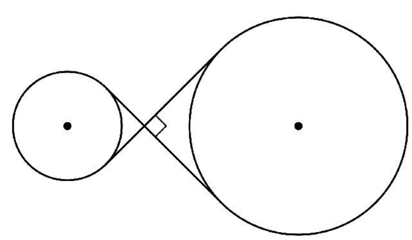
   (a) 70 (b) 90 (c) 110 (d) 120

4. A ferris wheel with radius 12 meters rotates clockwise at the rate of $10°$ per second. Initially, Richard is sitting on the seat at the very top of the wheel, which has a height of 27 meters above the ground. How high above the ground will Richard be after the wheel has rotated for exactly one minute?

   (a) 6 meters (c) $(3+4\sqrt{3})$ meters

   (b) 9 meters (d) $(15-6\sqrt{3})$ meters

5. The two internal tangents of two non-overlapping circles of radii 2 cm and 4 cm intersect at right angles, as shown in the figure below.

   What is the distance between the centers of the circles?

   (a) $6\sqrt{2}$ cm (b) $6\sqrt{3}$ cm (c) 12 cm (d) $12\sqrt{2}$ cm

6. How many positive integers less than or equal to 2015 are divisible by 3, but are neither divisible by 5 nor by 7?

   (a) 455 (b) 461 (c) 506 (d) 512

7. Which of the following is the fifth largest divisor of the number $2,015,000,000$?

   (a) $125,937,500$ (b) $155,000,000$ (c) $201,500,000$ (d) $251,875,000$

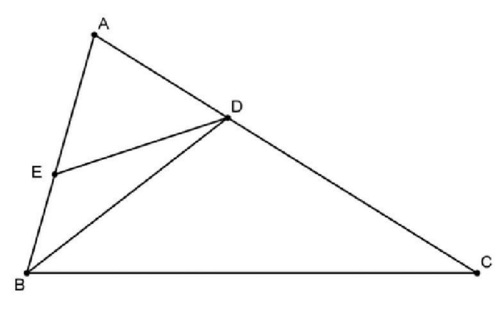
8. What is the remainder when $25^{2015}$ is divided by 18?

   (a) 11 (b) 13 (c) 15 (d) 17

9. Triangles $PQR$ and $QRS$, where $P \neq S$, are two right triangles sharing the hypotenuse $QR$. If $C_1$ is the circle passing through $P$, $Q$, and $R$, and $C_2$ is the circle passing through $Q$, $R$, and $S$, what can be said about $C_1 \cap C_2$?

   (a) $C_1 \cap C_2$ consists of exactly one point.
   (b) $C_1 \cap C_2$ consists of exactly two distinct points.
   (c) $C_1 \cap C_2$ is empty.
   (d) $C_1 \cap C_2$ consists of infinitely many points.

10. In triangle $ABC$, $BD$ is the angle bisector of $\angle ABC$. Moreover, $AB = BD$ and $AE = AD$. If $m\angle ACB = 36°$, determine $m\angle BDE$.




   (a) 24° (b) 18° (c) 15° (d) 12°

11. In how many ways can the letters of the word QUALIFYING be arranged such that the vowels are all in alphabetical order?

   (a) 24 (b) $151,200$ (c) $181,400$ (d) $3,628,800$

12. When $n+5$ is divided by 4, the remainder is 3. When $n+4$ is divided by 5, the remainder is 4. What is the remainder when $n+6$ is divided by 20?

(a) 4 (b) 5 (c) 14 (d) 16

13. It is known that $\frac{2}{5}$ of chips scattered on the table are red. We remove a quarter of the chips from the table, and it is found $\frac{1}{4}$ of those removed are red. What fraction of the chips that remained on the table are red?

(a) $\frac{27}{80}$ (b) $\frac{9}{20}$ (c) $\frac{1}{16}$ (d) $\frac{1}{6}$

14. Mary wants to give 15 cookies to Amy, Bob and Charlie. How many ways can she distribute the cookies to them such that Amy must receive at least 5 cookies while Bob and Charlie must each receive at least 1?

(a) 40 (b) 45 (c) 50 (d) 55

15. The exterior of a cube of side length 3 cm is painted red, and is then divided into 27 unit cubes. If one of these cubes is randomly selected and then rolled, what is the probability that a red face comes up?

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

**PART II.** Choose the best answer. Each correct answer is worth three points.

1. A married couple has PhP $50,000$ in their joint account. In anticipation of their upcoming anniversary, they agreed to split-up evenly and buy each one a gift. However, they agreed that they must have enough money left such that when combined, a minimum bank balance of PhP $5,000$ is maintained. If the couple individually bought gifts without regard to how much the other's gift will cost and each gift is randomly priced between PhP 0 to PhP $25,000$, what is the probability that they will be able to maintain the minimum required balance after buying the gifts?

(a) 2% (b) 20% (c) 80% (d) 98%

2. What is the difference between the largest and smallest real zeros of the function

$$f(x) = 2x^4 - 7x^3 + 2x^2 + 7x + 2?$$

(a) $\frac{5}{2}$ (b) $2\sqrt{2}$ (c) $1+\sqrt{2}$ (d) $\frac{3}{2}+\sqrt{2}$

3. Let $S$ be the set of all points $A$ on the circle $x^2 + (y-2)^2 = 1$ so that the tangent line at $A$ has a non-negative $y$-intercept; then $S$ is the union of one or more circular arcs. Find the total length of $S$.

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{3}$ (c) $\frac{5\pi}{4}$ (d) $\frac{7\pi}{6}$

4. How many positive integers $n$ make the expression

$$7^n + 7^3 + 2 \cdot 7^2$$

a perfect square?

(a) 0 (b) 1 (c) 3 (d) infinitely many

5. The numbers $1, 2, \ldots, 12$ have been arranged in a circle. In how many ways can five numbers be chosen from this arrangement so that no two adjacent numbers are selected?

(a) 15 (b) 36 (c) 180 (d) 396

6. How many (nonconstant) polynomial factors with leading coefficient 1, with the other coefficients possibly complex, does $x^{2015} + 18$ have?

(a) 2015 (c) $2^{2015}$

(b) $\sum_{k=1}^{2015} \binom{2015}{k}$ (d) $\sum_{k=0}^{2014} 2015^k$

7. Let $a = 25^{12}$, $b = 16^{14}$, and $c = 11^{16}$. Which of the following is true?

(a) $b < a < c$ (b) $a < c < b$ (c) $c < a < b$ (d) $b < c < a$

8. How many integers $x$ are there, where $100 \le x \le 2015$, and $x$ is divisible by 3 or 8, but not by 6?

(a) 400 (b) 402 (c) 479 (d) 481

9. Find the 2015th digit in 122333444455555...

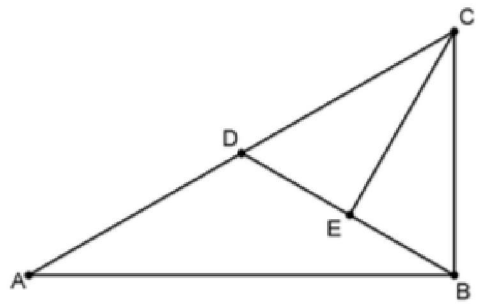
(a) 4 (b) 5 (c) 6 (d) 7

10. Find the sum of $\sum_{i=1}^{2015} \left\lfloor \frac{\sqrt{i}}{10} \right\rfloor$

(a) 5048 (b) 5050 (c) 5060 (d) 5064

**PART III.** All answers should be in simplest form. Each correct answer is worth six points.

1. In the right triangle $ABC$ where $m\angle B = 90°$, $BC : AB = 1 : 2$. Construct the median $BD$ and let point $E$ be on $BD$ such that $CE \perp BD$. Determine $BE : ED$.




2. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

   (a) $f(4xy) = 2y\,[f(x+y) + f(x-y)]$, for all $x, y \in \mathbb{R}$

   (b) $f(5) = 3$

   Find the value of $f(2015)$.

3. Let $P(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ be a polynomial with real coefficients and satisfying the property $P(n) = 10n$ for $n = 1, 2, 3, 4,$ and 5. Find the value of $a + b + c + d + e$.

4. Let $N = \{0, 1, 2, 3, \ldots\}$. Find the cardinality of the set

$$\{(a, b, c, d, e) \in N^5 : 0 \le a + b \le 2, 0 \le a + b + c + d + e \le 4\}.$$

5. How many terms are there when the expression of $(x+y+z)^{2015} + (x-y-z)^{2015}$ is expanded and simplified?

# $18^{\text{th}}$ Philippine Mathematical Olympiad

*Qualifying Stage*

17 October 2015

**PART I.** Choose the best answer. Each correct answer is worth two points.

1. D
2. B
3. A
4. C
5. D
6. B
7. D
8. B
9. C
10. B
11. A
12. D
13. C
14. C
15. D

**PART II.** Choose the best answer. Each correct answer is worth three points.

1. B
2. C
3. A
4. B
5. A
6. B
7. A
8. B
9. C
10. C
11. D
12. D
13. B
14. B
15. B

**PART III.** All answers should be in simplest form. Each correct answer is worth six points.

1. $2:3$
2. $\frac{1}{3}$
3. 1209
4. 120
5. $1,016,064$

# $18^{\text{th}}$ Philippine Mathematical Olympiad

*Area Stage*

14 November 2015

**PART I.** Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. Marc and Jon together have 66 marbles although Marc has twice as many marbles as Jon. Incidentally, Jon found a bag of marbles which enabled him to have three times as many marbles as Mark. How many marbles were in the bag that Jon found?

2. A camera's aperture determines the size of the circular opening in the lens that allows light in. If we want to allow twice as much light in, what should be the ratio of the new radius to the current radius?

3. Determine all values of $k \in \mathbb{R}$ for which the equation

$$\frac{4(2015^x) - 2015^{-x}}{2015^x - 3(2015^{-x})} = k$$

admits a real solution.

4. The points $(3, m)$, $(x_1, y_1)$ and $(x_2, y_2)$ are on the graph of the function $f(x) = \log_a x$. If $y_1 + y_2 = 2m$, find the value of $x_1 x_2$.

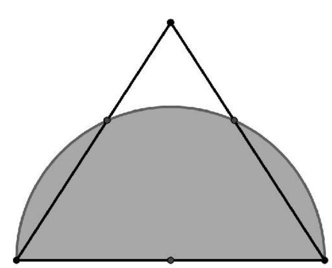
5. Let $a$, $b$, and $c$ be three consecutive even numbers such that $a > b > c$. What is the value of $a^2 + b^2 + c^2 - ab - bc - ac$?

6. Evaluate

$$\prod_{\theta=1}^{89} (\tan \theta^\circ \cos 1^\circ + \sin 1^\circ).$$

7. Find the sum of all the prime factors of 27,000,001.

8. Refer to the figure on the right. A side of an equilateral triangle is the diameter of the given semi-circle. If the radius of the semi-circle is 1, find the area of the unshaded region inside the triangle.



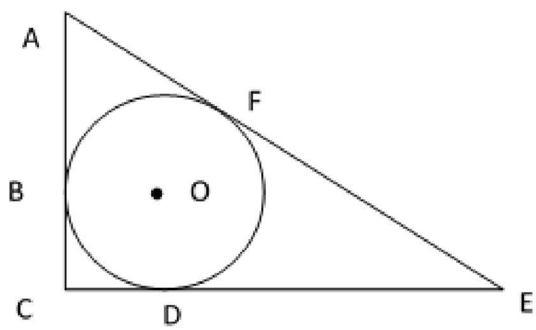
(Figure is not drawn to scale.)

9. How many ways can you place 10 identical balls in 3 baskets of different colors if it is possible for a basket to be empty?

10. Find the largest number $N$ so that

$$\sum_{n=5}^{N} \frac{1}{n(n-2)} < \frac{1}{4}.$$

11. Refer to the figure below. If circle $O$ is inscribed in the right triangle $ACE$ as shown below, and if the length of $AB$ is twice the length of $BC$, find the length of $CE$ if the perimeter of the right triangle is 36 units.




12. Find all real solutions to the system of equations

$$\begin{cases} x(y-1)+y(x+1) &= 6, \\ (x-1)(y+1) &= 1. \end{cases}$$

13. Find all real numbers $a$ and $b$ so that for all real numbers $x$,

$$2\cos^2\left(x+\frac{b}{2}\right) - 2\sin\left(ax-\frac{\pi}{2}\right)\cos\left(ax-\frac{\pi}{2}\right) = 1.$$

14. Let $P$ be the product of all prime numbers less than 90. Find the largest integer $N$ so that for each $n \in \{2, 3, 4, ..., N\}$, the number $P+n$ has a prime factor less than 90.

15. In how many ways can the letters of the word ALGEBRA be arranged if the order of the vowels must remain unchanged?

16. The lengths of the sides of a rectangle are all integers. Four times its perimeter is numerically equal to one less than its area. Find the largest possible perimeter of such a rectangle.

17. Find the area of the region bounded by the graph of $|x|+|y| = \frac{1}{4}|x+15|$.

18. Given $f(1-x)+(1-x)f(x) = 5$ for all real number $x$, find the maximum value that is attained by $f(x)$.

19. The amount 4.5 is split into two nonnegative real numbers uniformly at random. Then each number is rounded to its nearest integer. For instance, if 4.5 is split into $\sqrt{2}$ and $4.5-\sqrt{2}$, then the resulting integers are 1 and 3, respectively. What is the probability that the two integers sum up to 5?

20. Let $s_n$ be the sum of the digits of a natural number $n$. Find the smallest value of $\frac{n}{s_n}$ if $n$ is a four-digit number.

**PART II.** Show your solution to each problem. Each complete and correct solution is worth ten points.

1. The 6-digit number $739ABC$ is divisible by 7, 8, and 9. What values can $A$, $B$, and $C$ take?

2. The numbers from 1 to 36 can be written in a counterclockwise spiral as follows:

| | | | | | |
|---|---|---|---|---|---|
| $\boxed{31}$ | 30 | 29 | 28 | 27 | 26 |
| 32 | $\boxed{13}$ | 12 | 11 | 10 | 25 |
| 33 | 14 | $\boxed{3}$ | 2 | 9 | 24 |
| 34 | 15 | 4 | $\boxed{1}$ | 8 | 23 |
| 35 | 16 | 5 | 6 | $\boxed{7}$ | 22 |
| 36 | 17 | 18 | 19 | 20 | $\boxed{21}$ |

In the figure above, all the terms on the diagonal beginning from the upper left corner have been enclosed in a box, and these entries sum up to 76.

Suppose this spiral is continued all the way until 2015, leaving an incomplete square. Find the sum of all the terms on the diagonal beginning from the upper left corner of the resulting (incomplete) square.

3. Point $P$ on side $BC$ of triangle $ABC$ satisfies

$$|BP| : |PC| = 2 : 1.$$

Prove that the line $AP$ bisects the median of triangle $ABC$ drawn from vertex $C$.

# $18^{\text{th}}$ Philippine Mathematical Olympiad

*Area Stage*

14 November 2015

---

**PART I.** (3 points each)

1. 110 marbles
2. $\sqrt{2}$
3. $k \in (-\infty, \frac{1}{3}) \cup (4, +\infty)$
4. 9
5. 12
6. $\csc 1^\circ$ or $\sec 89^\circ$ or equivalent
7. 652
8. $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$
9. 66
10. 24
11. 12
12. $(4/3, 2), (-2, -4/3)$
13. $a = 1$ and $b = -\frac{3\pi}{2} + 2k\pi$, or $a = -1$ and $b = \frac{3\pi}{2} + 2k\pi$ where $k \in \mathbb{Z}$
14. 96
15. 840
16. 164 units
17. 30
18. 5
19. 4/9
20. $\dfrac{1099}{19}$

**PART II.** (10 points each)

1. Since $\gcd(7, 8, 9) = 1$, then $739ABC$ is divisible by $7, 8,$ and $9$ iff it is divisible by $7 \cdot 8 \cdot 9 = 504$. Note that the only integers between 739000 and 739999 which are divisible by 504 are 739368 and 739872. So, $(A, B, C) \in \{(3, 6, 8), (8, 7, 2)\}$.

2. Solution 1:

   The closest perfect square to 2015 is $2025 = 45^2$ which means that only the rightmost side will be incomplete while the required diagonal would still have a total of 45 entries.

   Looking at the values on the diagonal, we see that the numbers on the diagonal above 1 have a common second dierence. This suggests that this sequence satisfies a quadratic function of the form $f(n) = an^2 + bn + c$. Since $f(1) = 1$, $f(2) = 3$, $f(3) = 13$, solving a simple system of three equations gives us $f(n) = 4n^2 - 10n + 7$, $1 \le n \le 23$. On the other hand, the numbers on the diagonal below 1 also have a common second difference. This gives a sequence $g(n) = dn^2 + en + f$ with $g(1) = 1$, $g(2) = 7$, and $g(3) = 21$. By solving a similar system as above, we obtain $g(n) = 4n^2 - 6n + 3$, where $1 \le n \le 23$. Taking the sum of these two sequences of numbers, we have

$$\begin{aligned}
\sum_{n=1}^{23}[f(n)+g(n)] &= \sum_{n=1}^{23}\left(8n^2-16n+10\right)\\
&= 8\sum_{n=1}^{23}n^2-16\sum_{n=1}^{23}n+\sum_{n=1}^{23}10\\
&= 8\left[\frac{(23)(24)(47)}{6}\right]-168\left[\frac{(23)(24)}{2}\right]+10(23)\\
&= 30,406
\end{aligned}$$

Since 1 is counted twice, the required sum must be $30,406-1=30,405$.

Solution 2:

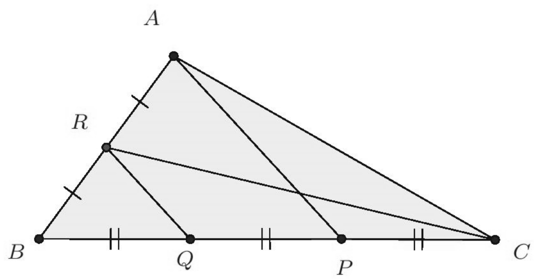
Filling the square with a few more numbers enables us to see that the boxed numbers

$$1,3,7,13,21,31,\dots,1981$$

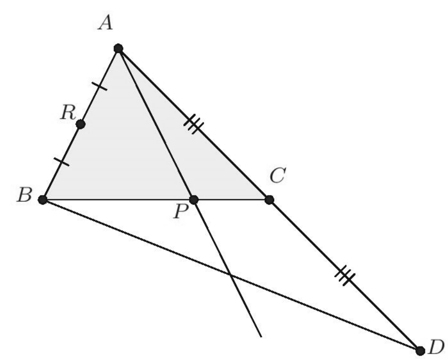
satisfy the recurrence relation $a_1=1$ and $(\forall n\in\mathbb{N})\ a_{n+1}=a_n+2n$. The associated homogeneous recurrence relation is solved by $a_n^{(h)}\equiv 1$. Testing a particular solution of the form $a_n^{(p)}=n(cn+d)$, we see that $c=1$ and $d=-1$. Therefore, the solution to the nonhomogeneous recurrence relation is $a_n=n^2-n+1$. The last boxed number 1981 corresponds to $n=45$. Therefore,

$$\sum_{n=1}^{45}(n^2-n+1)=\frac{45\cdot 46\cdot 91}{6}-\frac{45\cdot 46}{2}+45=30,405.$$

3. Solution 1: Let $Q$ be the midpoint of line segment $BP$. The conditions of the problem imply $|BQ|=|QP|=|PC|=\frac{1}{3}|BC|$. Let $R$ be the midpoint of line segment $AB$. Then $RQ$ is a midline of $ABP$. Consequently, $RQ\,||\,AP$. Ray $AP$ bisects side $CQ$ of triangle $CRQ$ while being parallel to side $RQ$ of this triangle. Thus $AP$ extends the midline of triangle $CRQ$ and bisects therefore also its side $CR$. But line segment $CR$ is the median of triangle $ABC$ drawn from vertex $C$.




Solution 2: Let $R$ be the midpoint of segment $AB$. Choose point $D$ on ray $AC$ beyond point $C$ such that $|AC|=|CD|$. Then $BC$ is a median of triangle $ABD$. As $|BP|:|PC|=2:1$, point $P$ is the intersection point of medians of triangle $ABD$. Thus $AP$ lies entirely on the other median of triangle $ABD$, i.e., ray $AP$ bisects the segment $BD$. As $CR$ is the midline of triangle $ABD$, we have $CR\,||\,BD$, implying that ray $AP$ also bisects the segment $CR$. But this is the median of triangle $ABC$ drawn from vertex $C$.



# MATHEMATICAL SOCIETY OF THE PHILIPPINES

***Promoting mathematics and mathematics education since 1973.***


2015 MSP Annual Convention, Laoag

| | |
|---|---|
| ***PRESIDENT*** | ***Jose Maria P. Balmaceda***<br>U.P. Diliman |
| ***VICE PRESIDENT*** | ***Emmanuel A. Cabral***<br>Ateneo de Manila University |
| ***SECRETARY*** | ***Kristine Joy E. Carpio***<br>De La Salle University |
| ***TREASURER*** | ***Marian P. Roque***<br>U.P. Diliman |
| ***MEMBERS*** | ***Fidel R. Nemenzo***<br>U.P. Diliman<br>***Jumela F. Sarmiento***<br>Ateneo de Manila University<br>***Maxima J. Acelajado***<br>De La Salle University<br>***Jose Ernie C. Lope***<br>U.P. Diliman<br>***Lorna S. Almocera***<br>U.P. Cebu |


SHARP
CALCULATORS
EL-W506X
X-series
SCIENTIFIC CALCULATOR
Fresh Looks Trendy Colors
Time -Tested Functions
Mathematics, Easier than ever!
PERSONAL CALCULATOR
EL-231L
Handy to have
around the house
even in your pockets!
EL-145T
OFFICE CALCULATOR
COVERING
various office needs
PRINTING CALCULATOR
EL-2607P
HARDWORKING
Calculator for ALL
Business Needs
Exclusively distributed by:
COLLINS INTERNATIONAL TRADING CORPORATION
412 Amang Rodriguez Ave., Manggahan, Pasig City
Tel nos.: (02) 681-6160/ 681-6161
Available in leading office supplies and book stores nationwide

*Make a mark through science.*
*Be part of the solution*

# Be a DOST Scholar.

**#Push4Science!**
**maging DOST Scholar ka**

**SCIENCE EDUCATION INSTITUTE**
**Department of Science and Technology**
2nd Level, Science Heritage Building, Sibol St.
DOST Compound, General Santos Avenue, Bicutan, Taguig City 1631
Phone Nos: (02) 839-0083, (02) 837-1333, (02) 837-2071 local 2382
Fax: (02) 839-0086
E-mail Address: seischolarships@gmail.com
Website: www.sei.dost.gov.ph
www.science-scholarships.ph
www.facebook.com/Push4Science