EASY 15 seconds, 2 points

1. The number $\overline{1ab76}$ is divisible by 72. List down all the possible values of $a+b$.

   Answer: 4 and 13 (accept 4 or 13 as well)

   Solution: Since $\overline{1ab76}$ is divisible by 72, it is divisible by 8 and 9. Since it is divisible by 9, $1+a+b+7+6$ or $14+a+b$ is divisible by 9. Therefore, $a+b$ is necessarily 4 or 13. The only other condition we require is that $\overline{b76}$ must be divisible by 8, which is equivalent to b being odd. Therefore $a+b$ being equal to 4 or 13 are both attainable.

2. A committee of three is to be selected from a pool of candidates consisting of five men and four women. If all the candidates are equally likely to be chosen, what is the probability that the committee will have an odd number of female members?

   Answer: $\dfrac{11}{21}$

   Solution: We either have exactly one or three female members. Therefore, the required probability is

$$\frac{\binom{4}{1}\binom{5}{2}+\binom{4}{3}}{\binom{9}{3}}=\frac{44}{84}=\frac{11}{21}.$$

3. The circle $(x+3)^2+(y-4)^2=50$ and the line $y=2x-5$ intersect at two points. Determine the distance between these two points.

   Answer: $2\sqrt{5}$

   Solution: From the given, we know $(x+3)^2+[(2x-5)-4]^2=50$.

$$\begin{aligned}(x+3)^2+(2x-9)^2&=50\\(x^2+6x+9)+(4x^2-36x+81)&=50\\5x^2-30x+40&=0\\5(x-2)(x-4)&=0\end{aligned}$$

   Thus, $x=2$ or $x=4$. If $x=2$, then $y=-1$. If $x=4$, then $y=3$. The distance between $(2,-1)$ and $(4,3)$ is $\sqrt{2^2+4^2}=2\sqrt{5}$.

4. In triangle ABC, we have $AB=BC=5$ and $CA=8$. What is the area of the region consisting of all points inside the triangle which are closer to AB than to AC?

   Answer: $\dfrac{60}{13}$

   Solution: Note that ABC is simply an isosceles triangle obtained by joining two 3-4-5 triangles; hence, its area is $3\cdot 4=12$. Then, if D is the foot of the angle bisector of $\angle A$ on BC, the area we want is actually just the area of ABD. By the angle bisector theorem,

$$\frac{[ABD]}{[ADC]}=\frac{AB}{AC}=\frac{5}{8},$$

   and so the area of ABD is $\frac{5}{13}$ that of ABC, that is, $\frac{5}{13}\cdot 12=\frac{60}{13}$.

5. If $\sin\theta+\cos\theta=\dfrac{6}{5}$, evaluate $\tan\theta+\cot\theta$.

   Answer: $\dfrac{50}{11}$

   Solution: We have

$$\tan\theta+\cot\theta=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}=\frac{1}{\sin\theta\cos\theta}.$$

Thus, since $\sin\theta+\cos\theta = \dfrac{6}{5}$, we have $(\sin\theta+\cos\theta)^2 = 1+2\sin\theta\cos\theta = \dfrac{36}{25}$, which gives $\sin\theta\cos\theta = \dfrac{11}{50}$. Hence, $\tan\theta+\cot\theta = \dfrac{50}{11}$.

6. The polynomial $p(x) = x^2 - 3x + 1$ has zeros r and s and a quadratic polynomial $q(x)$ has leading coefficient 1 and zeros r^3 and s^3. Find $q(1)$.

   Answer: -16

   Solution: We have $r + s = 3$ and $rs = 1$. By Vieta's formulas, we compute

   $$r^3 + s^3 = (r + s)^3 - 3rs(r + s) = 27 - 3(1)(3) = 18$$

   and $r^3s^3 = 1$. Thus, we obtain $q(x) = (x - r^3)(x - s^3) = x^2 - (r^3 + s^3)x + r^3s^3 = x^2 - 18x + 1$. Hence, we get $q(1) = -16$.

7. Find the smallest positive integer that is 20% larger than one integer and 19% smaller than another.

   Answer: 162

   Solution:
   Suppose N is our integer. Then we have $N = \frac{6}{5}x = \frac{81}{100}y$ for some integers x and y. In particular, x is divisible by 5 and y is divisible by 100. By multiplying by 100, we have $120x = 81y$, and the smallest integers that satisfy this as well as the previous conditions are $x = 135$, $y = 200$. This yields $N = 162$.

8. Find the number of ordered pairs (x, y) of positive integers satisfying $xy + x = y + 92$.

   Answer: 3

   Solution: We manipulate the equation to form $xy + x - y - 1 = 91$ or $(x - 1)(y + 1) = 91$. Since x and y must be positive integers, and $91 = 7 \times 13$, we must have

   $$\begin{cases} x-1=1 \\ y+1=91 \end{cases} \qquad \begin{cases} x-1=91 \\ y+1=1 \end{cases} \qquad \begin{cases} x-1=7 \\ y+1=13 \end{cases} \qquad \begin{cases} x-1=13 \\ y+1=7 \end{cases}$$

   Each of these systems of equations, except the second, yields a valid ordered pair (x, y), so the answer is $\boxed{3}$.

9. In square $ABCD$, P and Q are points on sides CD and BC, respectively, such that $\angle APQ = 90°$. If $AP = 4$ and $PQ = 3$, find the area of $ABCD$.

   Answer: $\dfrac{256}{17}$

   Solution: Note that triangles ADP and PCQ are similar, so $AD/PC = AP/PQ = 4/3$. Let $AD = 4x$ and $PC = 3x$. Since $ABCD$ is a square, $PD = x$. Applying Pythagorean theorem on triangle ADP, we have $x^2 + (4x)^2 = 16$, so that $x^2 = 16/17$. Hence, the area of square $ABCD$ is $16x^2 = 256/17$.

10. Determine the units digit of $(2! + 2)(3! + 3)(4! + 2)(5! + 3)\cdots(2018! + 2)(2019! + 3)$.

    Answer: 8

    Solution: In mod 10, the expression above becomes $(4)(9)(6)(2^{1007})(3^{1008}) = 2^{1010}\cdot 3^{1011}$ because starting from 5!, the units digit will be alternating from 3 and 2. In mod 10, $2^{1010} \cdot 3^{1011}$ becomes $(4)(7) = 28$, which ends with 8.

11. Andrew, Bob, and Chris are working together to finish a group project. If Andrew doesn't help, it would take 2 hours. If Bob doesn't help, it would take 3 hours. If Chris doesn't help, it would take 4 hours. Now they just learned that Dave can also help them out. If Dave works on the project alone, it would take him half a day. What is the least amount of time it would take for them to finish the group project?

    Answer: $\frac{8}{5}$ hours or 96 minutes

    Solution: Let A, B, C, and D be the number of hours Andrew, Bob, Chris, and Dave can finish the project when working alone. Then we have

$$\begin{aligned}\frac{1}{B}+\frac{1}{C}&=\frac{1}{2},\\ \frac{1}{A}+\frac{1}{C}&=\frac{1}{3},\\ \frac{1}{A}+\frac{1}{B}&=\frac{1}{4},\\ \frac{1}{D}&=\frac{1}{12}\end{aligned}$$

The least amount of time it would take to finish the group project would be achieved when all four work together on the project. This would be x hours, where

$$\begin{aligned}\frac{1}{x} &= \frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}\\ &= \frac{1}{2}\left[\left(\frac{1}{B}+\frac{1}{C}\right)+\left(\frac{1}{A}+\frac{1}{C}\right)+\left(\frac{1}{A}+\frac{1}{B}\right)\right]+\frac{1}{D}\\ &= \frac{15}{24}\end{aligned}$$

This means that $\frac{24}{15}=\frac{8}{5}$ hours or 96 minutes would be the least amount of time needed for the project.

12. PQR Entertainment wishes to divide their popular idol group PRIME, which consists of seven members, into three sub-units – PRIME-P, PRIME-Q, and PRIME-R – with each of these sub-units consisting of either two or three members. In how many different ways can they do this, if each member must belong to exactly one sub-unit?

    Answer: 630

    Solution: Note that the only way to do this is to divide PRIME into two sub-units of two members and one sub-unit of three members. There are $\binom{7}{3}\binom{4}{2}\binom{2}{2}=210$ ways to do this. Since there are 3 ways to choose which unit will have three members, our final answer is $210\cdot 3=630$.

13. Find the largest real number x such that $\sqrt[3]{x}+\sqrt[3]{4-x}=1$.

    Answer: $2+\sqrt{5}$

    Solution: Cubing both sides of the given equation yields $x+3\sqrt[3]{x(4-x)}(\sqrt[3]{x}+\sqrt[3]{4-x})+4-x=1$, which then becomes $4+3\sqrt[3]{x(4-x)}=1$ or $\sqrt[3]{x(4-x)}\overset{(1)}{=}-1$. Cubing both sides of equation (1) gives $x(4-x)=-1$ or $x^2-4x=1$. This means $(x-2)^2=5$ so $x=2\pm\sqrt{5}$ and the largest real solution is $x=2+\sqrt{5}$.

14. A function $f:\mathbb{R}\to\mathbb{R}$ satisfies

    $$f(x)+f(y)=f(x)f(y)+1-\frac{4}{xy}$$

    for all nonzero real numbers x and y. Given that $f(1)>0$, find the value of $f(4)$.

    Answer: $\frac{3}{2}$

    Solution: Setting $x=y=1$ in the functional equation and letting $c=f(1)>0$, we have $2c=c^2-3$ or $(c+1)(c-3)=0$. Thus $c=f(1)=3$ and setting $y=1$ yields

    $$f(x)+3=3f(x)+1-\frac{4}{x}\implies f(x)=1+\frac{2}{x}$$

    for all nonzero reals x. Hence, $f(4)=\frac{3}{2}$.

15. Last December 7, a computer owned by Patrick Laroche from Florida, USA determined that the number $2^{82,589,933} - 1$ is a prime number. This number had a whopping 24,862,048 digits, and is currently the largest known prime number. The computer used software provided by the GIMPS, which is a distributed computing project on the internet trying to find prime numbers of the form $2^p - 1$. What do you call such prime numbers?

    Answer: Mersenne primes

AVERAGE 45 seconds, 3 points

1. Basket A contains two white balls and three black balls, while Basket B contains a white ball and three black balls. Daniel randomly chooses one of the baskets and then randomly picks a ball from this basket. If he picked a white ball, what is the probability that his chosen basket was Basket A?

   Answer: $\dfrac{8}{13}$

   Solution: Let $P(A), P(B)$ be the probabilities of choosing Baskets A and B, respectively, and let $P(W)$ be the probability of picking a white ball. The probability of picking a white ball from Basket A is $P(W|A) = 2/5$ and the probability of picking a white ball from Basket B is $P(W|B) = 1/4$. With $P(A) = P(B) = 1/2$, we have $P(W) = P(A)P(W|A) + P(B)P(W|B) = (1/2)(2/5) + (1/2)(1/4) = 13/40$. Hence, the probability that a white ball Daniel picked came from the Basket A is

   $$P(A|W) = \frac{P(A)P(W|A)}{P(W)} = \frac{1/5}{13/40} = \frac{8}{13}.$$

2. The sum of the terms of an infinite geometric series is 2 and the sum of the squares of the terms of this series is 6. Find the sum of the cubes of the terms of this series.

   Answer: $\dfrac{96}{7}$

   Solution: Let a be the first term and let $r \in (-1, 1)$ be the common ratio of such infinite geometric series. Then, $a/(1-r) \overset{(1)}{=} 2$ and $a^2/(1-r^2) \overset{(2)}{=} 6$. Squaring (1) gives $a^2/(1-r)^2 = 4$ and using (2) yields $(1-r)/(1+r) = 3/2$. Solving for r, we get $r = -\frac{1}{5}$ so from (1) we get $a = 2(1-r) = \frac{12}{5}$. Thus,

   $$\frac{a^3}{1-r^3} = \frac{\frac{12^3}{5^3}}{1+\frac{1}{5^3}} = \frac{12^3}{5^3+1} = \frac{12 \cdot 12^2}{6(25-5+1)} = \frac{288}{21} = \frac{96}{7}.$$

3. How many positive perfect cubes are divisors of the product $1! \cdot 2! \cdot 3! \cdots 10!$?

   Answer: 468

   Solution: We have $N := 1! \cdot 2! \cdot 3! \cdots 10! = 2^{38}3^{17}5^{7}7^{4}$. Thus, a positive divisor of N that is a perfect cube must be of the form $2^{3a}3^{3b}5^{3c}7^{3d}$ for some nonnegative integers a, b, c, d. We see that $3a \leq 38, 3b \leq 17, 3c \leq 7$ and $3d \leq 4$. Thus, there are $\lceil \frac{38}{3} \rceil = 13$ choices for a, $\lceil \frac{17}{3} \rceil = 6$ choices for b, $\lceil \frac{7}{3} \rceil = 3$ choices for c, and $\lceil \frac{4}{3} \rceil = 2$ choices for d. Hence, there are $13 \cdot 6 \cdot 3 \times 2 = 468$ positive perfect cube divisors of N.

4. How many nonempty subsets of $\{1, 2, \ldots, 10\}$ have the property that the sum of its largest element and its smallest element is 11?

   Answer: 341

   Solution: If a is the smallest element of such a set, then $11 - a$ is the largest element, and for the remaining elements we may choose any (or none) of the $10 - 2a$ elements $a+1, a+2, \ldots, (11-a)-1$. Thus there are 2^{10-2a} such sets whose smallest element is a. We also require that $11 - a \geq a$ or $a < 6$. We see that there are

   $$\sum_{a=1}^{5} 2^{10-2a} = \sum_{a=0}^{4} 4^a = \frac{4^5 - 1}{4 - 1} = \frac{1023}{3} = 341$$

   possible sets.

5. Let $N = 2019^2 - 1$. How many positive factors of N^2 do not divide N?

Answer: 157

Solution: Note that $N = 2^3 \cdot 5 \cdot 101 \cdot 1009$ and so N has $4 \cdot 2 \cdot 2 \cdot 2 = 32$ factors. On the other hand, $N^2 = 2^6 \cdot 5^2 \cdot 101^2 \cdot 1009^2$ and so N^2 has $7 \cdot 3 \cdot 3 \cdot 3 = 189$ factors. Hence, $189 - 32 = 157$ of these factors of N^2 do not divide N.

6. Let a and b be integers such that in the expanded form of $(x^2 + ax + b)^3$, the numerical coefficient of x^4 is 99 and the numerical coefficient of x is 162. What are the values of a and b?

Answer: $a = 6, b = -3$

Solution: Using the multinomial expansion, the coefficients of x^4 and x are $3b + 3a^2$ and $3ab^2$, respectively. Thus, we just have to solve the system of equations below.

$$\begin{aligned} 3(b + a^2) &= 99 \\ 3ab^2 &= 162 \end{aligned}$$

Simplifying, we have $b + a^2 = 33$ and $ab^2 = 54$. Since the left sides of both equations are divisible by 3, then both a and b are divisible by 3. Let $b = 3n$ and $a = 3m$, where m and n are integers, then $3n + 9m^2 = 33$ and $(3m)(9n^2) = 54$. Simplifying, we have $n + 3m^2 = 11$ and $mn^2 = 2$. From these two equations, we can easily solve for m and n. This gives us $m = 2$ and $n = -1$. Thus, $a = 6$ and $b = -3$.

7. In trapezoid $ABCD$, AD is parallel to BC. If $AD = 52$, $BC = 65$, $AB = 20$, and $CD = 11$, find the area of the trapezoid.

Answer: 594

Solution: Extend AB and CD to intersect at E. Then $\sqrt{\frac{[EAD]}{[EBC]}} = \frac{AD}{BC} = \frac{4}{5} = \frac{EA}{EB} = \frac{ED}{EC}$. This tells us that $EB = 5AB = 100$, and $EC = 5CD = 55$. Triangle EBC has semiperimeter 110, and so by Heron's formula, the area of triangle EBC is given by $\sqrt{110(10)(55)(45)} = 1650$. Since $\frac{[AED]}{[BED]} = \frac{16}{25}$, the area of $ABCD$ is exactly $\frac{9}{25}$ of that, or 594.

8. A positive integer T is said to be *triangular* if $T = 1 + 2 + \cdots + n$ for some positive integer n. Find the smallest positive integer k such that whenever T is triangular, $81T + k$ is also triangular.

Answer: 10

Solution: Clearly, taking $T = 1$, we must have $k \geq 10$. We show that $k = 10$ indeed works. If T is triangular, then $T = \frac{n(n+1)}{2}$ for some positive integer n. Then, we have

$$\begin{aligned} 81T + 10 &= \frac{81n(n+1)}{2} + 10 \\ &= \frac{81n^2 + 81n + 20}{2} \\ &= \frac{(9n+4)(9n+5)}{2} \end{aligned}$$

which indeed is triangular as well.

9. Evaluate: $\sin 37^\circ \cos^2 34^\circ + 2 \sin 34^\circ \cos 37^\circ \cos 34^\circ - \sin 37^\circ \sin^2 34^\circ$.

Answer: $\dfrac{\sqrt{2+\sqrt{3}}}{2}$ or $\dfrac{\sqrt{6}+\sqrt{2}}{4}$

Solution: Note the following:

$$\begin{aligned} \sin 37^\circ \cos^2 34^\circ + \sin 34^\circ \cos 37^\circ \cos 34^\circ &= \cos 34^\circ (\sin 37^\circ \cos 34^\circ + \sin 34^\circ \cos 37^\circ) \\ &= \cos 34^\circ \sin(37^\circ + 34^\circ) \\ &= \cos 34^\circ \sin 71^\circ \end{aligned}$$

$$
\begin{aligned}
\sin 34^\circ \cos 37^\circ \cos 34^\circ - \sin 37^\circ \sin^2 34^\circ &= \sin 34^\circ (\cos 37^\circ \cos 34^\circ - \sin 37^\circ \sin 34^\circ) \\
&= \sin 34^\circ \cos(37^\circ + 34^\circ) \\
&= \sin 34^\circ \cos 71^\circ
\end{aligned}
$$

Thus, the required sum is $\cos 34^\circ \sin 71^\circ + \sin 34^\circ \cos 71^\circ = \sin 105^\circ = \sin 75^\circ = \dfrac{\sqrt{6}+\sqrt{2}}{4}$.

10. A triangle has sides of lengths 20 and 19. If the triangle is not acute, how many possible integer lengths can the third side have?

    Answer: 16

    Solution: We begin by determining the possible integer lengths of the third side s. By the triangle inequality, $2 \le s \le 38$.

    Now, we determine when the triangle is not acute. This means that the triangle is either a right triangle or an obtuse triangle. Since we are only considering integer lengths, notice that the third side will always be the shortest or the longest side. These are precisely the two cases to consider.

    We use the Pythagorean Theorem to determine when the triangle is right or obtuse. If the two shorter sides of the triangle are a and b and the longest side is c, then the triangle is right or obtuse when $a^2 + b^2 \le c^2$.

    If the third side is the shortest, then by the above inequality we have that $s^2 \le 20^2 - 19^2$. This gives us $s^2 \le 39$. This in turn means that $2 \le s \le 6$ for this case, a total of 5 possible lengths of the third side.

    If the third side is the longest, then by the above inequality we have that $19^2 + 20^2 \le s^2$. This gives us $s^2 \ge 761$. This in turn means that $28 \le s \le 38$ for this case, or a total of 11 possible lengths of the third side.

    Thus, there are $5 + 11 = 16$ possible lengths of the third side.

DIFFICULT 90 seconds, 6 points

1. Let $\triangle ABC$ be a right triangle with right angle at B. Let the points D, E, and F be on AB, BC, and CA, respectively, such that $\triangle DEF$ is an equilateral triangle and $EC = FC$. If $DB = 5\sqrt{3}, BE = 3$, and $\sin \angle ACB = 4\sqrt{3}/7$, find the perimeter of $\triangle ADF$.

   Answer: $35\sqrt{3} + 63 + 2\sqrt{21}$

   Solution: By Pythagorean Theorem, $DE = \sqrt{(5\sqrt{3})^2 + 3^2} = \sqrt{84} = EF$. From $\sin \angle ACB = 4\sqrt{3}/7$, we have $\cos \angle ACB = 1/7$. Let $EC = FC = x$, then by Cosine Law on side EF of $\triangle ECF$, we have $EF^2 = x^2 + x^2 - 2x^2 \cos \angle ACB$. Solving for x^2, we have $x^2 = 84/[2(1-1/7)] = 49$. Hence, $EC = FC = x = 7$ and $BC = BE + EC = 10$. Since $\cos \angle ACB = 1/7$, then $AC = 70$, which means $AF = 63$. Since $\sin \angle ACB = 4\sqrt{3}/7$, then $AB = 40\sqrt{3}$, which means $AD = 35\sqrt{3}$. Thus, the perimeter of $\triangle ADF$ is $35\sqrt{3} + 63 + 2\sqrt{21}$.

2. Gari took a 6-item multiple choice test with 3 choices per item, labelled A, B, and C. After the test, he tried to recall his answers to the items. He only remembered that he never answered three consecutive A's, he never answered three consecutive B's, and he did not leave any item blank. How many possible sets of answers could Gari have had?

   Answer: 569

   Solution: The problem is equivalent to that of finding the number of ternary strings of length 6 that do not contain any 3 consecutive 0's or 3 consecutive 1's. Using the principle of inclusion and exclusion, it suffices to count the number of ternary strings of length 6 that contain at least 3 consecutive 0's, multiply the number by 2 (by symmetry), and then subtract the number of ternary strings of length 6 that contain at least 3 consecutive 0s and 3 consecutive 1s.

Note that there are 60 such strings with exactly 3 consecutive 0s, 16 with exactly 4 consecutive 0's, 4 with exactly 5 consecutive 0's, and 1 with exactly 6 0's. This gives a total of $60+16+4+1=81$ strings of length 6 that contain at least 3 consecutive 0's.

The total number of ternary strings of length 6 is 3^6 while the number of ternary strings of length 6 that contain both 3 consecutive 0s and 3 consecutive 1s is 2. Thus, the number of strings that satisfy the problem are $3^6-(2)(81)+2=569$.

3. Let $x=-\sqrt{2}+\sqrt{3}+\sqrt{5}, y=\sqrt{2}-\sqrt{3}+\sqrt{5}$, and $z=\sqrt{2}+\sqrt{3}-\sqrt{5}$. What is the value of the expression below?

$$\frac{x^4}{(x-y)(x-z)}+\frac{y^4}{(y-z)(y-x)}+\frac{z^4}{(z-x)(z-y)}$$

Answer: 20

Solution: Writing the expression as a single fraction, we have

$$\frac{x^4y-xy^4+y^4z-yz^4-zx^4+xz^4}{(x-y)(x-z)(y-z)}$$

Note that if $x=y, x=z$, or $y=z$, then the numerator of the expression above will be 0. Thus, $(x-y)(x-z)(y-z)|(x^4y-xy^4+y^4z-yz^4-zx^4+xz^4)$. Moreover, the numerator can be factored as follows.

$$(x-y)(x-z)(y-z)(x^2+y^2+z^2+xy+yz+zx)$$

Hence, we are only evaluating $x^2+y^2+z^2+xy+yz+zx$, which is equal to

$$\begin{aligned}\frac{1}{2}\left[(x+y)^2+(y+z)^2+(z+x)^2\right] &= \frac{1}{2}\left[(2\sqrt{5})^2+(2\sqrt{2})^2+(2\sqrt{3})^2\right]\\ &= 2(5+2+3)\\ &= 20\end{aligned}$$

4. In acute triangle ABC, M and N are the midpoints of sides AB and BC, respectively. The tangents to the circumcircle of triangle BMN at M and N meet at P. Suppose that AP is parallel to BC, $AP=9$ and $PN=15$. Find AC.

Answer: $20\sqrt{2}$

Solution: Extend rays PM and CB to meet at Q. Since $AP \parallel QC$ and M is the midpoint of AB, triangles AMP and BMQ are congruent. This gives $QM=MP=PN=15$ and $QB=AP=9$. Observe that the circumcircle of triangle BMN is tangent to QM, so by power of a point, we compute $QM^2=QB\cdot QN$, which yields $QN=25$. Applying Stewart's theorem on triangle QNP, we have

$$15\cdot 15^2+15\cdot 25^2=30\cdot 15\cdot 15+30MN^2,$$

which implies that $MN^2=\frac{1}{2}(25^2-15^2)=200$. Hence, as MN is the midline of triangle ABC, we obtain $AC^2=4MN^2=800$ and $AC=20\sqrt{2}$.

5. For each positive integer n, let $\varphi(n)$ be the number of positive integers from 1 to n that are relatively prime to n. Evaluate

$$\sum_{n=1}^{\infty}\frac{\varphi(n)4^n}{7^n-4^n}.$$

Answer: $\frac{28}{9}$

Solution: We compute

$$\sum_{n=1}^{\infty}\frac{\varphi(n)4^n}{7^n-4^n}=\sum_{n=1}^{\infty}\varphi(n)\frac{(\frac{4}{7})^n}{1-(\frac{4}{7})^n}=\sum_{n=1}^{\infty}\varphi(n)\sum_{k=1}^{\infty}\left(\frac{4}{7}\right)^{nk}.$$

Interchanging the order of summation and using the fact that $\sum_{d|n}\varphi(d)=n$ where the sum takes all over positive divisors of n, we arrive at

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\varphi(n)4^n}{7^n-4^n} &= \sum_{n=1}^{\infty}\varphi(n)\sum_{k=1}^{\infty}\left(\frac{4}{7}\right)^{nk} = \sum_{m=1}^{\infty}\left(\frac{4}{7}\right)^{m}\sum_{d|m}\varphi(d)\\ &= \sum_{m=1}^{\infty}m\left(\frac{4}{7}\right)^{m} = \frac{\frac{4}{7}}{(1-\frac{4}{7})^2} = \frac{28}{9}.\end{aligned}$$

SPARE

1. (Easy) Among all victims of zombie bites, 10% are prescribed the experimental drug Undetenin to treat them. Overall, 4% of the human population suffer an adverse reaction to Undetenin. Out of all the patients being treated with Undetenin, 2% suffer an adverse reaction to the drug. What is the probability that a patient allergic to Undetenin is prescribed the drug?

   Answer: 5%

   Solution: This is an application of Bayes' theorem. If A is the event of being prescribed Undetenin and B is being allergic to it, then we are looking for $P(A|B)$. We have $P(A|B)=\frac{P(B|A)P(A)}{P(B)}=\frac{(0.02)(0.1)}{0.04}=0.05$.

2. (Average) For the upcoming semester, 100 math majors can take up to two out of five math electives. Suppose 22 will not take any math elective in the coming semester. Also,

   - 7 will take Algebraic Number Theory and Galois Theory
   - 12 will take Galois Theory and Hyperbolic Geometry
   - 3 will take Hyperbolic Geometry and Cryptography
   - 15 will take Cryptography and Topology
   - 8 will take Topology and Algebraic Number Theory.

   Everyone else will take only one math elective. Furthermore, 16 will take either Algebraic Number Theory or Cryptography, but not both. How many math majors will take exactly one of Galois Theory, Hyperbolic Geometry, or Topology?

   Answer: 17

   Solution: We solve this problem by eliminating those math majors who do not fit the necessary criteria of taking exactly one of Galois Theory, Hyperbolic Geometry, or Topology. The 22 math majors who are not taking any math electives are immediately excluded. Also eliminated are the 7 taking ANT and GT, the 12 taking GT and HG, and the 3 taking HG and C, because they are not taking GT or HG exclusively. We also eliminate the 16 taking either ANT only or C only. Finally, we subtract those who are not taking Topology exclusively. Thus, we arrive at the answer to the problem

   $$100-22-7-12-3-16-15-8=17.$$

3. (Difficult) A geometric sequence with at least three terms and a rational common ratio has first term 32^{16} and last term 625^{30}. If the product of all possible values of the second term of this sequence is $16^x\cdot 625^y$, where x and y are integers, what are the values of x and y?

   Answer: $x=214$ and $y=69$

   Solution: Let a be the common ratio of the geometric sequence and k be the number of terms, then $32^{16}\cdot a^{k-1}=625^{30}$ or $2^{80}\cdot a^{k-1}=5^{120}$. Solving for a, we have

   $$a=\sqrt[k-1]{\frac{5^{120}}{2^{80}}}=\left(\frac{5^{120}}{2^{80}}\right)^{1/(k-1)}.$$

From the equation above, $k-1$ must divide both 80 and 120. If $k-1$ is even, then we will consider the positive and negative values of the $(k-1)$th root. Since the $\text{GCF}(80, 120) = 40$, then the possible values of $k-1$ are $2, 4, 5, 8, 10, 20, 40$ (not 1 because there will only be two terms in the sequence). Thus, there are 13 possible values of a, which are

$$\pm\frac{5^{60}}{2^{40}}, \pm\frac{5^{30}}{2^{20}}, \frac{5^{24}}{2^{16}}, \pm\frac{5^{15}}{2^{10}}, \pm\frac{5^{12}}{2^{8}}, \pm\frac{5^{6}}{2^{4}}, \pm\frac{5^{3}}{2^{2}}$$

Thus, the product of all possible second terms of the sequence is

$$(2^{80})^{13}\left(\frac{5^{120+60+24+30+24+12+6}}{2^{80+40+16+20+16+8+4}}\right) = 2^{1040}\left(\frac{5^{276}}{2^{184}}\right) = 2^{856}\cdot 5^{276} = 16^{214}\cdot 625^{69}.$$

Therefore, $x = 214$ and $y = 69$.