22nd Philippine Mathematical Olympiad

Qualifying Stage, 12 October 2019

PART I. Choose the best answer. Each correct answer is worth two points.

1. If $2^{x-1}+2^{x-2}+2^{x-3}=\dfrac{1}{16}$, find 2^x.

(a) $\dfrac{1}{14}$ (b) $\dfrac{2}{3}$ (c) $\sqrt[14]{2}$ (d) $\sqrt[3]{4}$

2. If the number of sides of a regular polygon is decreased from 10 to 8, by how much does the measure of each of its interior angles decrease?

(a) $30°$ (b) $18°$ (c) $15°$ (d) $9°$

3. Sylvester has 5 black socks, 7 white socks, 4 brown socks, where each sock can be worn on either foot. If he takes socks randomly and without replacement, how many socks would be needed to guarantee that he has at least one pair of socks of each color?

(a) 13 (b) 14 (c) 15 (d) 16

4. Three dice are simultaneously rolled. What is the probability that the resulting numbers can be arranged to form an arithmetic sequence?

(a) $\dfrac{1}{18}$ (b) $\dfrac{11}{36}$ (c) $\dfrac{7}{36}$ (d) $\dfrac{1}{6}$

5. Sean and the bases of three buildings A, B, and C are all on level ground. Sean measures the angles of elevation of the tops of buildings A and B to be $62°$ and $57°$, respectively. Meanwhile, on top of building C, CJ spots Sean and determines that the angle of depression of Sean from his location is $31°$. If the distance from Sean to the bases of all three buildings is the same, arrange buildings A, B, and C in order of increasing heights.

(a) C, B, A (b) B, C, A (c) A, C, B (d) A, B, C

6. A function $f:\mathbb{R}\to\mathbb{R}$ satisfies $f(xy)=f(x)/y^2$ for all positive real numbers x and y. Given that $f(25)=48$, what is $f(100)$?

(a) 1 (b) 2 (c) 3 (d) 4

7. A trapezoid has parallel sides of lengths 10 and 15; its two other sides have lengths 3 and 4. Find its area.

(a) 24 (b) 30 (c) 36 (d) 42

8. Find the radius of the circle tangent to the line $3x + 2y + 4 = 0$ at $(-2, 1)$ and whose center is on the line $x - 8y + 36 = 0$.

(a) $2\sqrt{13}$ (b) $2\sqrt{10}$ (c) $3\sqrt{5}$ (d) $5\sqrt{2}$

9. A circle is inscribed in a rhombus which has a diagonal of length 90 and area 5400. What is the circumference of the circle?

(a) 36π (b) 48π (c) 72π (d) 90π

10. Suppose that n identical promo coupons are to be distributed to a group of people, with no assurance that everyone will get a coupon. If there are 165 more ways to distribute these to four people than there are ways to distribute these to three people, what is n?

(a) 12 (b) 11 (c) 10 (d) 9

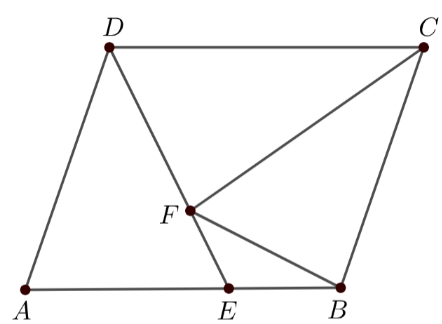
11. Let x and y be positive real numbers such that

$$\log_x 64 + \log_{y^2} 16 = \frac{5}{3} \qquad \text{and} \qquad \log_y 64 + \log_{x^2} 16 = 1.$$

What is the value of $\log_2(xy)$?

(a) 16 (b) 3 (c) $\frac{1}{3}$ (d) $\frac{1}{48}$

12. The figure below shows a parallelogram $ABCD$ with $CD = 18$. Point F lies inside $ABCD$ and lines AB and DF meet at E. If $AE = 12$ and the areas of triangles FEB and FCD are 30 and 162, respectively, find the area of triangle BFC.



(a) 162 (b) 156 (c) 150 (d) 144

13. A *semiprime* is a natural number that is the product of two primes, not necessarily distinct. How many subsets of the set $\{2, 4, 6, \ldots, 18, 20\}$ contain at least one semiprime?

(a) 768 (b) 896 (c) 960 (d) 992

14. The number whose base-b representation is 91_b is divisible by the number whose base-b representation is 19_b. How many possible values of b are there?

(a) 2 (b) 3 (c) 4 (d) 5

15. The number of ordered pairs (a, b) of relatively prime positive integers such that $ab = 36!$ is

(a) 128 (b) 1024 (c) 2048 (d) 4096

PART II. Choose the best answer. Each correct answer is worth three points.

16. Which of the following cannot be the difference between a positive integer and the sum of its digits?

(a) 603 (b) 684 (c) 765 (d) 846

17. Evaluate the sum

$$\sum_{n=0}^{2019} \cos\left(\frac{n^2\pi}{3}\right).$$

(a) 0 (b) 1 (c) -1 (d) $\frac{1}{2}$

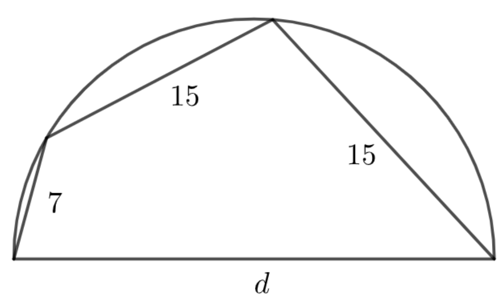
18. There is an unlimited supply of red 4×1 tiles and blue 7×1 tiles. In how many ways can an 80×1 path be covered using nonoverlapping tiles from this supply?

(a) 2381 (b) 3382 (c) 5384 (d) 6765

19. For a real number t, $\lfloor t \rfloor$ is the greatest integer less than or equal to t. How many natural numbers n are there such that $\left\lfloor \frac{n^3}{9} \right\rfloor$ is prime?

(a) 3 (b) 9 (c) 27 (d) infinitely many

20. A quadrilateral with sides of lengths 7, 15, 15, and d is inscribed in a semicircle with diameter d, as shown in the figure below.



Find the value of d.

(a) 18 (b) 22 (c) 24 (d) 25

21. Find the sum of all real numbers b for which all the roots of the equation $x^2 + bx - 3b = 0$ are integers.

(a) 4 (b) -8 (c) -12 (d) -24

22. A number x is selected randomly from the set of all real numbers such that a triangle with side lengths 5, 8, and x may be formed. What is the probability that the area of this triangle is greater than 12?

(a) $\dfrac{3\sqrt{15}-5}{10}$ (b) $\dfrac{3\sqrt{15}-\sqrt{41}}{10}$ (c) $\dfrac{3\sqrt{17}-5}{10}$ (d) $\dfrac{3\sqrt{17}-\sqrt{41}}{10}$

23. Two numbers a and b are chosen randomly from the set $\{1, 2, \ldots, 10\}$ in order, and with replacement. What is the probability that the point (a, b) lies above the graph of $y = ax^3 - bx^2$?

(a) $\dfrac{4}{25}$ (b) $\dfrac{9}{50}$ (c) $\dfrac{19}{100}$ (d) $\dfrac{1}{5}$

24. For a real number t, $\lfloor t \rfloor$ is the greatest integer less than or equal to t. How many integers n are there with $4 \le n \le 2019$ such that $\lfloor \sqrt{n} \rfloor$ divides n and $\lfloor \sqrt{n+1} \rfloor$ divides $n+1$?

(a) 44 (b) 42 (c) 40 (d) 38

25. The number $20^5 + 21$ has two prime factors which are three-digit numbers. Find the sum of these numbers.

(a) 1112 (b) 1092 (c) 1062 (d) 922

PART III. All answers should be in simplest form. Each correct answer is worth six points.

26. Find the number of ordered triples of integers (m, n, k) with $0 < k < 100$ satisfying

$$\frac{1}{2^m} - \frac{1}{2^n} = \frac{3}{k}.$$

27. Triangle ABC has $\angle BAC = 60^\circ$ and circumradius 15. Let O be the circumcenter of ABC and let P be a point inside ABC such that $OP = 3$ and $\angle BPC = 120^\circ$. Determine the area of triangle BPC.

28. A string of 6 digits, each taken from the set $\{0, 1, 2\}$, is to be formed. The string should <u>not</u> contain any of the substrings 012, 120, and 201. How many such 6-digit strings can be formed?

29. Suppose a, b, and c are positive integers less than 11 such that

$$\begin{aligned} 3a + b + c &\equiv abc \pmod{11} \\ a + 3b + c &\equiv 2abc \pmod{11} \\ a + b + 3c &\equiv 4abc \pmod{11} \end{aligned}$$

What is the sum of all the possible values of abc?

30. Find the minimum value of $\dfrac{7x^2 - 2xy + 3y^2}{x^2 - y^2}$ if x and y are positive real numbers such that $x > y$.

Answers

Part I. (2 points each)

1. A
2. D
3. B
4. C
5. A
6. C
7. B
8. A
9. C
10. D
11. A
12. D
13. C
14. B
15. C

Part II. (3 points each)

16. B
17. A
18. C
19. A
20. D
21. D
22. C
23. C
24. B
25. A

Part III. (6 points each)

26. 13
27. $54\sqrt{3}$
28. 492
29. 198
30. $2\sqrt{6} + 2$