21st Philippine Mathematical Olympiad

Qualifying Stage, 20 October 2018

PART I. Choose the best answer. Each correct answer is worth two points.

1. The measures of the angles of a pentagon form an arithmetic sequence with common difference $15°$. Find the measure of the largest angle.

   (a) $78°$ (b) $103°$ (c) $138°$ (d) $153°$

2. If $x - y = 4$ and $x^2 + y^2 = 5$, find the value of $x^3 - y^3$.

   (a) -24 (b) -2 (c) 2 (d) 8

3. Five numbers are inserted between 4 and 2916 so that the resulting seven numbers form a geometric sequence. What is the the fifth term of this geometric sequence?

   (a) 324 (b) 416 (c) 584 (d) 972

4. The constant term in the expansion of $\left(3x^2 - \frac{1}{x}\right)^6$ is

   (a) 189 (b) 135 (c) 90 (d) 54

5. Juan has 4 distinct jars and a certain number of identical balls. The number of ways that he can distribute the balls into the jars such that each jar has at least one ball is 56. How many balls does he have?

   (a) 9 (b) 8 (c) 7 (d) 6

6. A regular octagon of area 48 is inscribed in a circle. If a regular hexagon is inscribed in the same circle, what would its area be?

   (a) $12\sqrt{10}$ (b) $18\sqrt{6}$ (c) $24\sqrt{3}$ (d) $30\sqrt{2}$

7. What is the smallest positive integer which when multiplied to $24^4 + 64$ makes the product a perfect square?

   (a) 1037 (b) 2074 (c) 5185 (d) 10370

8. A bowl of negligible thickness is in the shape of a truncated circular cone, with height 4 in and upper and lower radii of 9 in and 6 in, respectively. What is the volume of the bowl?

   (a) 276π in^3 (b) 248π in^3 (c) 234π in^3 (d) 228π in^3

9. A circle is tangent to the line $2x - y + 1 = 0$ at the point $(2, 5)$ and the center is on the line $x + y - 9 = 0$. Find the radius of the circle.

(a) $\sqrt{14}$ (b) 4 (c) $3\sqrt{2}$ (d) $2\sqrt{5}$

10. Suppose that 16 points are drawn on a plane such that exactly 7 of these points are collinear. Any set of three points which do not all belong to the 7 are noncollinear. If 3 random points are selected from the 16 points, what is the probability that a triangle can be formed by joining these points?

(a) $\frac{15}{16}$ (b) $\frac{17}{20}$ (c) $\frac{19}{20}$ (d) $\frac{63}{80}$

11. The points $(0, -1)$, $(1, 1)$, and (a, b) are distinct collinear points on the graph of $y^2 = x^3 - x + 1$. Find $a + b$.

(a) -6 (b) -2 (c) 1 (d) 8

12. What is the probability that a positive divisor of $2^{20}3^{17}$ also divides $2^8 3^6$?

(a) $\frac{12}{85}$ (b) $\frac{2}{9}$ (c) $\frac{1}{9}$ (d) $\frac{1}{6}$

13. Let ABC be a right triangle where $AB = 7$, $BC = 24$, and with hypotenuse AC. Point D is on AC such that $AD : DC = 2 : 3$. Let m and n be the relatively prime positive integers such that $BD^2 = \frac{m}{n}$. What is $m + n$?

(a) 554 (b) 550 (c) 544 (d) 540

14. In chess, a knight moves by initially taking two steps in any horizontal or vertical direction and then taking one more step in any direction that is perpendicular to its initial movement. Suppose Renzo places a knight on a random tile on an 8×8 chessboard. Find the probability that he can land on a corner tile in exactly two moves.

(a) $\frac{3}{16}$ (b) $\frac{1}{4}$ (c) $\frac{9}{16}$ (d) $\frac{5}{8}$

15. In rectangle $ABCD$, point Q lies on side AB such that $AQ : QB = 1 : 2$. Ray CQ is extended past Q to R so that AR is parallel to BD. If the area of triangle ARQ is 4, what is the area of rectangle $ABCD$?

(a) 108 (b) 120 (c) 132 (d) 144

PART II. Choose the best answer. Each correct answer is worth three points.

1. How many two-digit numbers are there such that the product of their digits is equal to a prime raised to a positive integer exponent?

(a) 27 (b) 28 (c) 29 (d) 30

2. $ABCDEF$ is a six-digit perfect square in base 10 such that $DEF = 8 \times ABC$. What is $A + B + C + D + E + F$? (Note that $ABCDEF$, ABC, and DEF should be interpreted as numerals in base 10 and not as products of digits.)

   (a) 18 (b) 27 (c) 36 (d) 45

3. Quadrilateral $ABCD$ has $AB = 25$, $BC = 60$, $CD = 39$, $DA = 52$, and $AC = 65$. What is the inradius of $\triangle BCD$?

   (a) 14 (b) 15 (c) 16 (d) 18

4. Find the sum of the first 20 positive integers that are multiples of either 3 or 7 but not both.

   (a) 336 (b) 399 (c) 529 (d) 592

5. Q is a rational function with $xQ(x + 2018) = (x - 2018)Q(x)$ for all $x \notin \{0, 2018\}$. If $Q(1) = 1$, what is $Q(2017)$?

   (a) 2020 (b) 2019 (c) 2018 (d) 2017

6. How many ordered pairs (x, y) of positive integers are there such that $1 \leq x \leq y \leq 20$ and both $\frac{y}{x}$ and $\frac{y+2}{x+2}$ are integers?

   (a) 38 (b) 36 (c) 34 (d) 32

7. An entertainment agency has seven trainees. Each of the trainees does at least one of dancing, singing, and rapping, and no two trainees have the same skill set. How many ways can the agency choose three trainees to form a group, provided that the group must have at least one dancer, one singer, and one rapper (who are not necessarily distinct)?

   (a) 26 (b) 29 (c) 32 (d) 35

8. Let $\triangle ABC$ be a right triangle such that its hypotenuse AC has length 10 and $\angle BAC = 15°$. Let O be the center of the circumcircle of $\triangle ABC$. Let E be the point of intersection of the lines tangent to the circumcircle at points A and B, and F be the point of intersection of the lines tangent to the circumcircle at points B and C. The area of $\triangle OEF$ is

   (a) $\frac{25}{8}$ (b) 50 (c) 100 (d) $\frac{225}{8}$

9. A real number x is chosen randomly from the interval $(0, 1)$. What is the probability that $\lfloor \log_5(3x) \rfloor = \lfloor \log_5 x \rfloor$? (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

   (a) $\frac{1}{6}$ (b) $\frac{5}{36}$ (c) $\frac{1}{9}$ (d) $\frac{1}{12}$

10. What is the remainder when $1^{2018} + 2^{2018} + \cdots + 2017^{2018}$ is divided by 2018?

   (a) 0 (b) 2 (c) 1009 (d) 2017

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. Suppose two numbers are randomly selected in order, and without replacement, from the set $\{1, 2, 3, \ldots, 888\}$. Find the probability that the difference of their squares is not divisible by 8.

2. Let $P(x)$ be a polynomial with degree 2018 whose leading coefficient is 1. If $P(n) = 3n$ for $n = 1, 2, \ldots, 2018$, find $P(-1)$.

3. A sequence $\{a_n\}_{n\geq 1}$ of positive integers is defined by $a_1 = 2$ and for integers $n > 1$,
$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} + \frac{n}{a_n} = 1.$$
Determine the value of $\displaystyle\sum_{k=1}^{\infty} \frac{3^k(k+3)}{4^k a_k}$.

4. In triangle ABC, D and E are points on sides AB and AC respectively, such that BE is perpendicular to CD. Let X be a point inside the triangle such that $\angle XBC = \angle EBA$ and $\angle XCB = \angle DCA$. If $\angle A = 54^\circ$, what is the measure of $\angle EXD$?

5. Define $g : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ and $h : \mathbb{R} \setminus \{\sqrt{3}\} \to \mathbb{R}$ as follows:
$$g(x) = \frac{1+x}{1-x} \qquad \text{and} \qquad h(x) = \frac{\sqrt{3}+3x}{3-\sqrt{3}x}.$$
How many ways are there to choose $f_1, f_2, f_3, f_4, f_5 \in \{g, h\}$, not necessarily distinct, such that $(f_1 \circ f_2 \circ f_3 \circ f_4 \circ f_5)(0)$ is well-defined and equal to 0?

Answers

Part I. (2 points each)

1. C
2. B
3. A
4. B
5. A
6. B
7. C
8. D
9. D
10. A
11. D
12. D
13. A
14. D
15. B

Part II. (3 points each)

1. C
2. B
3. A
4. C
5. D
6. D
7. C
8. B
9. A
10. C

Part III. (6 points each)

1. $\frac{5}{8}$
2. $2019! - 3$
3. $\frac{21}{8}$
4. 36°
5. 8