# 21st Philippine Mathematical Olympiad

*Area Stage*, 24 November 2018

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**PART I.** Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. How many distinct prime factors does $5^{14} - 30 + 5^{13}$ have?

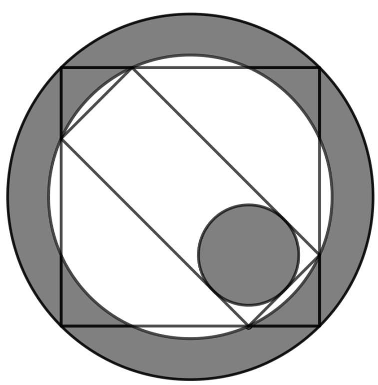
2. Given that $a$ and $b$ are real numbers satisfying the equation
$$\log_{16} 3 + 2\log_{16}(a-b) = \frac{1}{2} + \log_{16} a + \log_{16} b,$$
find all possible values of $\frac{a}{b}$.

3. Find the minimum value of the expression $\sqrt{(x-1)^2 + (y+1)^2} + \sqrt{(x+3)^2 + (y-2)^2}$.

4. In how many ways can the letters of the word COMBINATORICS be arranged so that the letters C, O, A, C, T, O, R, S appear in that order in the arrangement (although there may be letters in between)?

5. Let $N$ be the smallest positive integer divisible by 20, 18, and 2018. How many positive integers are both less than and relatively prime to $N$?

6. A square is inscribed in a circle, and a rectangle is inscribed in the square. Another circle is circumscribed about the rectangle, and a smaller circle is tangent to three sides of the rectangle, as shown below. The shaded area between the two larger circles is eight times the area of the smallest circle, which is also shaded. What fraction of the largest circle is shaded?



7. In $\triangle ABC$, the length of $AB$ is 12 and its incircle $O$ has radius 4. Let $D$ be the point of tangency of circle $O$ with $AB$. If $AD : AB = 1 : 3$, find the area of $\triangle ABC$.

8. Suppose that $\{a_n\}_{n\geq 1}$ is an arithmetic sequence of real numbers such that
$$a_1 + a_2 + a_3 + a_4 + \cdots + a_{10} = 20,$$
$$a_1 + a_4 + a_9 + a_{16} + \cdots + a_{100} = 18.$$
Compute $a_1 + a_8 + a_{27} + a_{64} + \cdots + a_{1000}$.

9. Let $\alpha$ and $\beta$ be the roots of the equation $x^2 - 11x + 24 = 0$. Let $f$ be the polynomial of least degree, with integer coefficients and leading coefficient 1, such that $\sqrt{\alpha} + \sqrt{\beta}$ and $\sqrt{\alpha\beta}$ are zeros of $f$. Find $f(1)$.

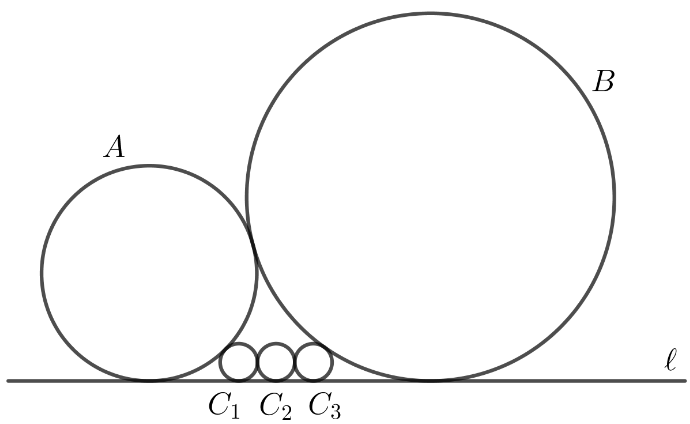
10. Suppose that the lengths of the sides of a right triangle are integers and its area is six times its perimeter. What is the least possible length of its hypotenuse?

11. A *Vitas word* is a string of letters that satisfies the following conditions:

    - It consists of only the letters B, L, R.
    - It begins with a B and ends in an L.
    - No two consecutive letters are the same.

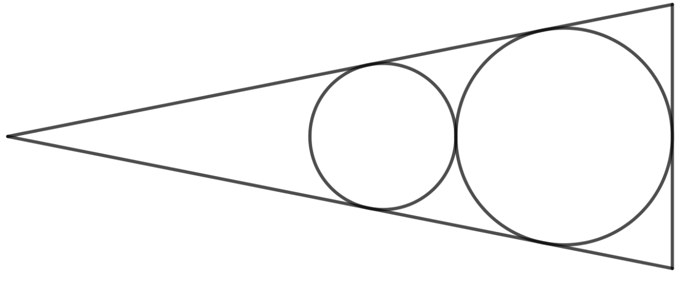
    How many Vitas words are there with 11 letters?

12. In the figure below, five circles are tangent to line $\ell$. Each circle is externally tangent to two other circles. Suppose that circles $A$ and $B$ have radii 4 and 225, respectively, and that $C_1, C_2, C_3$ are congruent circles. Find their common radius.




13. Let $S = \{1, 2, 3, \ldots, 12\}$. Find the number of *nonempty* subsets $T$ of $S$ such that if $x \in T$ and $3x \in S$, then it follows that $3x \in T$.

14. In the figure below, the incircle of the isosceles triangle has radius 3. The smaller circle is tangent to the incircle and the two congruent sides of the triangle. If the smaller circle has radius 2, find the length of the base of the triangle.



15. Evaluate the expression $(1 + \tan 7.5^\circ)(1 + \tan 18^\circ)(1 + \tan 27^\circ)(1 + \tan 37.5^\circ)$.

16. Compute the number of ordered 6-tuples $(a, b, c, d, e, f)$ of positive integers such that

$$a + b + c + 2(d + e + f) = 15.$$

17. Let $S = \{1, 2, \ldots, 2018\}$. For each subset $T$ of $S$, take the product of all elements of $T$, with 1 being the product corresponding to the empty set. The sum of all such resulting products (with repetition) is $N$. Two elements $m$ and $n$ of $S$, with $m < n$, are removed. The sum of all products over all subsets of the resulting set is $\dfrac{N}{2018}$. What is $n$?

18. Let $\alpha$ be the unique positive root of the equation

$$x^{2018} - 11x - 24 = 0.$$

Find $\lfloor \alpha^{2018} \rfloor$. (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$.)

19. How many distinct numbers are there in the sequence $\left\lfloor \dfrac{1^2}{2018} \right\rfloor, \left\lfloor \dfrac{2^2}{2018} \right\rfloor, \ldots, \left\lfloor \dfrac{2018^2}{2018} \right\rfloor$?

20. Suppose that $a, b, c$ are real numbers such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 4\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) = \frac{c}{a+b} + \frac{a}{b+c} + \frac{b}{c+a} = 4.$$

Determine the value of $abc$.

**PART II.** Show your solution to each problem. Each complete and correct solution is worth ten points.

1. For a positive integer $n$, let $\varphi(n)$ denote the number of positive integers less than and relatively prime to $n$. Let $S_k = \displaystyle\sum_n \frac{\varphi(n)}{n}$, where $n$ runs through all positive divisors of $42^k$. Find the largest positive integer $k < 1000$ such that $S_k$ is an integer.

2. In $\triangle ABC$, $AB > AC$ and the incenter is $I$. The incircle of the triangle is tangent to sides $BC$ and $AC$ at points $D$ and $E$, respectively. Let $P$ be the intersection of the lines $AI$ and $DE$, and let $M$ and $N$ be the midpoints of sides $BC$ and $AB$, respectively. Prove that $M$, $N$, and $P$ are collinear.

3. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ satisfying, for all $n \in \mathbb{N}$,

   (a) $|f(n)| = n$

   (b) $0 \leq \displaystyle\sum_{k=1}^{n} f(k) < 2n$.

   Evaluate $\displaystyle\sum_{n=1}^{2018} f(n)$.

**Answers to the 21st PMO Area Stage**

**Part I. (3 points each)**

1. 7
2. 3
3. 5
4. 77220
5. 48384
6. $\dfrac{9}{25}$
7. 96
8. 2
9. $-92$
10. 58
11. 341
12. $\dfrac{9}{4}$
13. 1151
14. $3\sqrt{6}$
15. 4
16. 119
17. 1008
18. 35
19. 1514
20. $\dfrac{49}{23}$

**Part II. (10 points each, full solutions required)**

1. For a positive integer $n$, let $\varphi(n)$ denote the number of positive integers less than and relatively prime to $n$. Let $S_k = \sum_n \frac{\varphi(n)}{n}$, where $n$ runs through all positive divisors of $42^k$. Find the largest positive integer $k < 1000$ such that $S_k$ is an integer.

   Answer: 996

   Solution: The function $\varphi$ is the well-known Euler totient function which satisfies the property

$$\frac{\varphi(n)}{n} = \prod_{\substack{p \mid n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)$$

for any integer $n > 2$. Note that the problem defines $\varphi(1) = 0$.

For any $k \in \mathbb{N}$, the number $42^k$ has $(k+1)^3$ factors, each of which takes the form $2^a 3^b 7^c$ where $a, b, c \in \{0, 1, 2, \ldots, k\}$. Since $\varphi(n)/n$ depends only on the prime factors of $n$, we partition this set of factors into 8 forms with the same value for $\varphi(n)/n$.

| | form of $n$ | number of such $n$s | $\frac{\varphi(n)}{n}$ | contribution to the sum |
|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 0 |
| 2 | $2^a;\ a = 1, 2, \ldots, k$ | $k$ | $\left(1 - \frac{1}{2}\right) = \frac{1}{2}$ | $\frac{k}{2}$ |
| 3 | $3^b;\ b = 1, 2, \ldots, k$ | $k$ | $\left(1 - \frac{1}{3}\right) = \frac{2}{3}$ | $\frac{2k}{3}$ |
| 4 | $7^c;\ c = 1, 2, \ldots, k$ | $k$ | $\left(1 - \frac{1}{7}\right) = \frac{6}{7}$ | $\frac{6k}{7}$ |
| 5 | $2^a 3^b;\ a, b = 1, 2, \ldots, k$ | $k^2$ | $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = \frac{1}{3}$ | $\frac{k^2}{3}$ |
| 6 | $2^a 7^c;\ a, c = 1, 2, \ldots, k$ | $k^2$ | $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{7}\right) = \frac{3}{7}$ | $\frac{3k^2}{7}$ |
| 7 | $3^b 7^c;\ b, c = 1, 2, \ldots, k$ | $k^2$ | $\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right) = \frac{4}{7}$ | $\frac{4k^2}{7}$ |
| 8 | $2^a 3^b 7^c,\ a, b, c = 1, 2, \ldots, k$ | $k^3$ | $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{7}\right) = \frac{2}{7}$ | $\frac{2k^3}{7}$ |

Therefore, $S_k = \frac{k}{2} + \frac{2k}{3} + \frac{6k}{7} + \frac{k^2}{3} + \frac{3k^2}{7} + \frac{4k^2}{7} + \frac{2k^3}{7} = \frac{a(k)}{42}$, where $a(k) = 85k + 56k^2 + 12k^3$. Hence, the problem wants us to find the largest $k < 10^3$ so that $a(k) \equiv 0 \pmod{42}$, or equivalently, $a(k) \equiv 0 \pmod{2}$, $a(k) \equiv 0 \pmod{3}$, and $a(k) \equiv 0 \pmod{7}$. Observe that

- $a(k) \equiv k \pmod{2}$, which is 0 iff $k$ is even.
- $a(k) \equiv k + 2k^2 \pmod{3}$, which is 0 iff $k \equiv 0$ or $1 \pmod{3}$
- $a(k) \equiv k + 5k^3 \pmod{7}$, which is 0 iff $k \equiv 0, 2,$ or $5 \pmod{7}$.

The numbers 999 and 997 are not even. $998 \equiv 2 \pmod{3}$. 996 is even, $\equiv 0 \pmod{3}$, and $\equiv 2 \pmod{7}$. Therefore, the answer is 996.

2. In $\triangle ABC$, $AB > AC$ and the incenter is $I$. The incircle of the triangle is tangent to sides $BC$ and $AC$ at points $D$ and $E$, respectively. Let $P$ be the intersection of the lines $AI$ and $DE$, and let $M$ and $N$ be the midpoints of sides $BC$ and $AB$, respectively. Prove that $M$, $N$, and $P$ are collinear.

Solution: Let $\alpha = \angle A$, $\beta = \angle B$, and $\gamma = \angle C$. We will show that $\angle BNP = \angle BNM$.

*Claim*: Points $B$, $I$, $D$, and $P$ are concyclic.

*Proof of Claim*:

Since $\triangle DCE$ is isosceles with $CD = CE$,

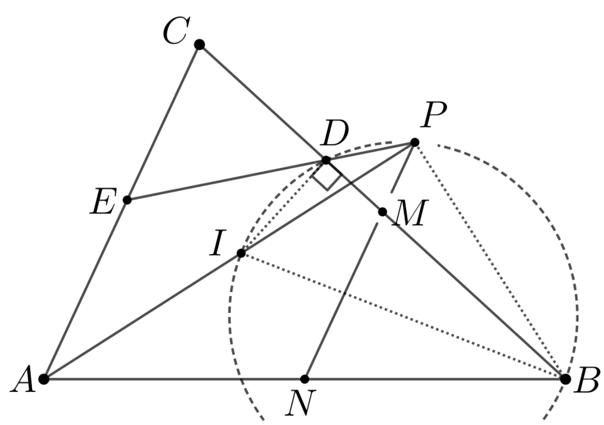
$$\angle BDP = \angle CDE = \frac{1}{2}(180^\circ - \gamma) = \frac{1}{2}(\alpha + \beta).$$

On the other hand,

$$\angle BIP = \angle BAI + \angle IBA = \frac{\alpha}{2} + \frac{\beta}{2}.$$

Therefore, $\angle BDP = \angle BIP$. Thus, $B$, $I$, $D$, and $P$ are concyclic.

Alternative: $\angle DPI = \angle EPA = \angle CED - \angle EAP = \frac{1}{2}(180^\circ - \gamma) - \frac{1}{2}\alpha = \frac{1}{2}\beta = \angle DBI$. Thus, $B$, $I$, $D$, and $P$ are concyclic. ■




This implies that since $ID \perp BC$, then $\angle APB = \angle IPB = \angle IDB = 90^\circ$. In right $\triangle APB$, $N$ is the midpoint of the hypotenuse so it follows that $NP = NA$.

Consequently, $\angle BNP = \angle NAP + \angle APN = 2\angle NAP = \alpha$. Alternatively, since $\angle APB = 90^\circ$, then $AB$ is a diameter of the circumcircle of $\triangle APB$, and $N$ is the circumcenter. Consequently, $\angle BNP = 2\angle BAP = \alpha$.

Since $M$ and $N$ are the midpoints of $AB$ and $CB$ respectively, $\angle BNM = \angle BAC = \alpha$.

We now have $\angle BNP = \angle BNM$. Therefore, $P$, $M$, and $N$ are collinear.

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Alternative Approaches via the introduction of a phantom point

- Extend $AI$ and $NM$ to meet at $P'$. Goal: Show $P = P'$ by showing that $P'$ is on line $ED$.
- Extend $ED$ and $NM$ to meet at $P''$. Goal: Show $P = P''$ by showing that $P''$ is on line $AI$.

3. Consider the function $f : \mathbb{N} \to \mathbb{Z}$ satisfying, for all $n \in \mathbb{N}$,

   (a) $|f(n)| = n$

   (b) $0 \leq \sum_{k=1}^{n} f(k) < 2n$.

   Evaluate $\sum_{n=1}^{2018} f(n)$.

   Answer: 2649

   Solution: Let $S_n = \sum_{k=1}^{n} f(k)$. We want the value of $S_{2018}$.

   Claim: $f(n) = \begin{cases} n & \text{if } S_{n-1} < n \\ -n & \text{if } S_{n-1} \geq n \end{cases}$

   Proof: The inequality condition is $0 \leq S_{n-1} + f(n) < 2n$.

   - If $n > S_{n-1}$, then $0 \leq S_{n-1} + f(n) < n + f(n)$ so $f(n) > -n$. Therefore, $f(n) = n$.
   - If $n \leq S_{n-1}$, then $n + f(n) \leq S_{n-1} + f(n) < 2n$ so $f(n) < n$. Therefore, $f(n) = -n$.

   Claim: If $S_n = 0$, then

| $S_{n+1}$ | $S_{n+2}$ | $S_{n+3}$ | $S_{n+4}$ | $\cdots$ | $S_{n+2j}$ | $S_{n+2j+1}$ | $\cdots$ |
|---|---|---|---|---|---|---|---|
| $n+1$ | $2n+3$ | $n$ | $2n+4$ | $\cdots$ | $2n+2+j$ | $n+1-j$ | $\cdots$ |

   where $j = 1, 2, \ldots, n+1$.

   The pattern here: $S_{n+1}, S_{n+3}, S_{n+5}, \ldots$ are numbers decreasing by 1, while $S_{n+2}, S_{n+4}, S_{n+6}, \ldots$ are numbers increasing by 1.

   Proof: $S_n = 0 < n$ so $f(n+1) = n+1$. Thus, $S_{n+1} = 0 + (n+1) = n+1$.

   $S_{n+1} = n+1 < n+2$ so $f(n+2) = n+2$. Thus, $S_{n+2} = n+1+(n+2) = 2n+3$.

   $S_{n+2} = 2n+3 > n+3$ so $f(n+3) = -n-3$. Thus, $S_{n+3} = 2n+3+(-n-3) = n$.

   We prove the claim by strong induction. Suppose the pattern holds for $S_{n+1}, S_{n+2}, \ldots, S_{n-1+2j}$.

   Since $S_{n-1+2j} = S_{n+1+2(j-1)} = n+1-(j-1) = n+2-j < n+2j$, then $f(n+2j) = n+2j$ so $S_{n+2j} = (n+2-j)+(n+2j) = 2n+2+j$.

   On the other hand, since $S_{n+2j} = 2n+2+j = (n+2j+1)+(n+1-j) \geq n+2j+1$, then $f(n+2j+1) = -(n+2j+1)$ so $S_{n+2j+1} = (2n+2+j)-(n+2j+1) = n+1-j$, which proves the claim.

   Eventually, $S_{n+1}, S_{n+3}, \ldots, S_{n+2j+1}, \ldots$ will decrease to 0, when $j = n+1$. Thus, if $S_n = 0$, it follows that the next 0 value is $S_{3(n+1)}$.

   Therefore, $S_3 = 0$, $S_{3\cdot 4} = S_{12} = 0$, $S_{3\cdot 13} = S_{39} = 0$, $S_{3\cdot 40} = S_{120} = 0$, $S_{3\cdot 121} = S_{363} = 0$, $S_{3\cdot 364} = S_{1092} = 0$.

   Since $2018 = 1092 + 2\cdot 463$, then $S_{2018} = 2\cdot 1092 + 2 + 463 = 2649$.