**20th Philippine Mathematical Olympiad**
**20 January 2018**
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EASY 15 seconds, 2 points

1. A line passes through $(k, -9)$ and $(7, 3k)$ and has slope $2k$. Find the possible values of $k$.

   Answer: $\dfrac{9}{2}$ or 1

   Solution: We have $\frac{3k-(-9)}{7-k} = 2k$, so $3k + 9 = 2k(7 - k)$ or $2k^2 - 11k + 9 = (2k - 9)(k - 1) = 0$. Hence $k = \frac{9}{2}$ or $k = 1$.

2. Let $\Delta ABC$ be a right triangle with legs $AB = 6$ and $BC = 8$. Let $R$ and $r$ be the circumradius and the inradius of $\Delta ABC$, respectively. Find the sum of $R$ and $r$.

   Answer: 7

   Solution: Since $\Delta ABC$ is a right triangle, its circumradius is just half of its hypotenuse. By Pythagorean theorem, $AC = \sqrt{6^2 + 8^2} = 10$ and so $R = \frac{AC}{2} = 10/2 = 5$. Now, we can compute the inradius by using the following identity:

   $$\left( r \times \frac{AB + BC + AC}{2} = \text{ Area } (\Delta ABC) \right).$$

   Since $\Delta ABC$ is a right triangle, its area is just $(AB \times BC)/2 = 48/2 = 24$. So that

   $$r = \frac{24}{(6 + 8 + 10)/2} = 2.$$

   Thus, $R + r = 5 + 2 = 7$.

3. In how many ways can the letters of the word $CHIEF$ be arranged such that $I$ appears at some position after $E$?

   Answer: 60

   Solution: Start with an empty string that should have 5 letters. We first choose where to put $I$ and $E$, and there are $\binom{5}{2}$ ways of doing this. Each way has 3! ways to rearrange the remaining letters, so the number of ways is $10 * 6 = 60$ ways.

   Alternatively, note that the letters can be arranged without restriction in 5! ways. Since there are the same number of arrangements where $I$ appears before $E$ as those where $E$ appears before $I$, the required number is $5!/2 = 60$ ways.

4. We say that the constant $a$ is a fixed point of a function $f$ if $f(a) = a$. Find all values of $c$ such that $f(x) = x^2 - 2$ and $g(x) = 2x^2 - c$ share a common fixed point.

   Answer: 3 and 6

   Solution: Note that if $f(x) = x$, then $x^2 - x - 2 = 0$, or $(x - 2)(x + 1) = 0$. Thus, the fixed points of $f$ are 2 and $-1$, and so we wish for either of these to be a fixed point of $g$. If 2 is a fixed point of $g$, we must have $g(2) = 2 = 8 - c$, and so $c = 6$. If $-1$ is a fixed point of $g$, then $2 - c = -1$, and $c = 3$.

5. How many pairs of positive integers $(a, b)$ are there, both not exceeding 10, such that $a \leq \gcd(a, b) \leq b$ and $a \leq \text{lcm}(a, b) \leq b$?

   Answer: 27

   Solution: $\gcd(a, b) \geq a \implies \gcd(a, b) = a$. Likewise, $\text{lcm}(a, b) = b$. This can only happen if $b$ is a multiple of $a$. Hence the answer is $10 + 5 + 3 + 2 + 2 + 1 + 1 + 1 + 1 + 1 = 27$.

6. Suppose that $\log_a 125 = \log_5 3$ and $\log_b 16 = \log_4 7$. Find the value of $a^{(\log_5 3)^2} - b^{(\log_4 7)^2}$.

   Answer: $-22$

   Solution: Note that $a^{\log_5 3} = 125$ and $b^{\log_4 7} = 16$. Thus

$$a^{(\log_5 3)^2} - b^{(\log_4 7)^2} = (a^{\log_5 3})^{\log_5 3} - (b^{\log_4 7})^{\log_4 7} = 125^{\log_5 3} - 16^{\log_4 7} = 3^3 - 7^2 = -22.$$

7. Two cards are chosen, without replacement, from a deck of 50 cards numbered $1, 2, 3, \ldots, 50$. What is the probability that the product of the numbers on these cards is divisible by 7?

   Answer: $\dfrac{46}{175}$

   Solution:
   Note that there are $\lfloor \frac{50}{7} \rfloor = 7$ multiples of 7 from 1 to 50, so there are 43 numbers which are not divisible by 7. The probability that after choosing two cards, the product of these numbers is not divisible by 7 is $\binom{43}{2}/\binom{50}{2} = \frac{43\cdot 42}{50\cdot 49} = \frac{129}{175}$. Hence, the probability in question is then $1 - \frac{129}{175} = \frac{46}{175}$.

8. A triangle with side lengths 24, 70, 74 is inscribed in a circle. Find the difference between the numerical values of the area and the circumference of the circle in terms of $\pi$.

   Answer: $1295\pi$

   Solution: Notice that the given triangle is a right triangle with hypotenuse equal to 74 because $12^2 + 35^2 = 37^2$ which implies that $24^2 + 70^2 = 74^2$. From this, we know that the area and circumference of the circle are $37^2(\pi)$ or $1369\pi$ and $74\pi$, respectively. Thus, the difference is equal to $1295\pi$.

9. Find the smallest positive real numbers $x$ and $y$ such that $x^2 - 3x + 2.5 = \sin y - 0.75$.

   Answer: $x = \dfrac{3}{2}, y = \dfrac{\pi}{2}$

   Solution: The equation above can be rewritten as $\left(x - \frac{3}{2}\right)^2 + 1 = \sin y$. Since the left side is always greater or equal to 1 and the right side is always less than or equal to 1, then both sides must be equal to 1. This means $x = \frac{3}{2}$ and $y = (4k+1)\frac{\pi}{2}$ for all $k \in \mathbb{Z}$. Choosing $k = 0$ gives the smallest possible value of $y$.

10. A standard deck of 52 cards has the usual 4 suits and 13 denominations. What is the probability that two cards selected at random, and without replacement, from this deck will have the same denomination or have the same suit?

    Answer: $\dfrac{5}{17}$

    Solution: Let $A$ be the event that the 2 chosen cards will have the same denomination; and let $B$ be the event that the 2 chosen cards will have the same suit. Note that $A \cap B = \emptyset$. So that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$.

    Since there are 4 suits to choose from, then there are ${}_4C_2 = 6$ possible pairs of the same value. There are a total of 13 card values, so that

$$\mathbb{P}(A) = \frac{(13)(6)}{{}_{52}C_2} = \frac{1}{17}.$$

    Since there are 13 card denominations to choose from, then there are ${}_{13}C_2 = 78$ possible pairs of the same suit. There are a total of 4 suits, so that

$$\mathbb{P}(B) = \frac{(4)(78)}{{}_{52}C_2} = \frac{4}{17}.$$

    Thus, $\mathbb{P}(A \cup B) = \dfrac{1+4}{17} = \dfrac{5}{17}$.

11. Let $a$ and $b$ be integers for which $\dfrac{a}{2} + \dfrac{b}{1009} = \dfrac{1}{2018}$. Find the smallest possible value of $|ab|$.

Answer: 504

Solution: Clear denominators to write this as $1009a + 2b = 1$. Clearly, $a = 1$, $b = -504$ is a solution, and so our solutions are of the form $a = 1 + 2k$, $b = -504 - 1009k$. Now, clearly $|a| \geq 1$, and $|b| \geq 504$, so $|ab| \geq 504$, and equality is attained when $a = 1$ and $b = -504$.

12. Let $x$ be a real number satisfying $x^2 - \sqrt{6}x + 1 = 0$. Find the numerical value of $\left|x^4 - \dfrac{1}{x^4}\right|$.

Answer: $8\sqrt{3}$

Solution: Note that $x + \frac{1}{x} = \sqrt{6}$. Then

$$\begin{aligned}
\left|x^4 - \frac{1}{x^4}\right| &= \left(x^2 + \frac{1}{x^2}\right)\left|x^2 - \frac{1}{x^2}\right| \\
&= \left(x^2 + \frac{1}{x^2}\right)\left(x + \frac{1}{x}\right)\left|x - \frac{1}{x}\right| \\
&= \left(\left(x + \frac{1}{x}\right)^2 - 2\right)\left(x + \frac{1}{x}\right)\left(\sqrt{\left(x + \frac{1}{x}\right)^2 - 4}\right) \\
&= 4\left(\sqrt{6}\right)\left(\sqrt{2}\right) = 8\sqrt{3}
\end{aligned}$$

13. Factor $(a+1)(a+2)(a+3)(a+4) - 120$ completely into factors with integer coefficients.

Answer: $(a^2 + 5a + 16)(a-1)(a+6)$

Solution: We have

$$\begin{aligned}
(a+1)(a+2)(a+3)(a+4) - 120 &= (a+1)(a+4)(a+2)(a+3) - 120 \\
&= (a^2 + 5a + 4)(a^2 + 5a + 6) - 120 \\
&= (a^2 + 5a + 5)^2 - 1 - 120 = (a^2 + 5a + 5)^2 - 121 \\
&= (a^2 + 5a + 5 + 11)(a^2 + 5a + 5 - 11) \\
&= (a^2 + 5a + 16)(a^2 + 5a - 6) \\
&= (a^2 + 5a + 16)(a - 1)(a + 6).
\end{aligned}$$

14. Suppose there are 3 distinct green balls, 4 distinct red balls, and 5 distinct blue balls in an urn. The balls are to be grouped into pairs such that the balls in any pair have different colors. How many sets of six pairs can be formed?

Answer: 1440

Solution: Since there are 7 green and red balls combined, and 5 blue balls, then there should be one pair with 1 green and 1 red, and all the other pairs must have a blue ball. There are $3 \times 4$ ways of selecting a green-red pair, and $5!$ ways of selecting the partners for the blue balls. Thus, there are $3 \times 4 \times 5! = 1440$ possible pairings.

15. This year, our country's team will be participating in the 59th International Mathematical Olympiad, to be held in Cluj-Napoca, Romania. The IMO, which was first held in 1959 also in Romania, has been held annually except in 1980, when it was cancelled due to internal strife in its host country. Which East Asian country was supposed to host the 1980 IMO?

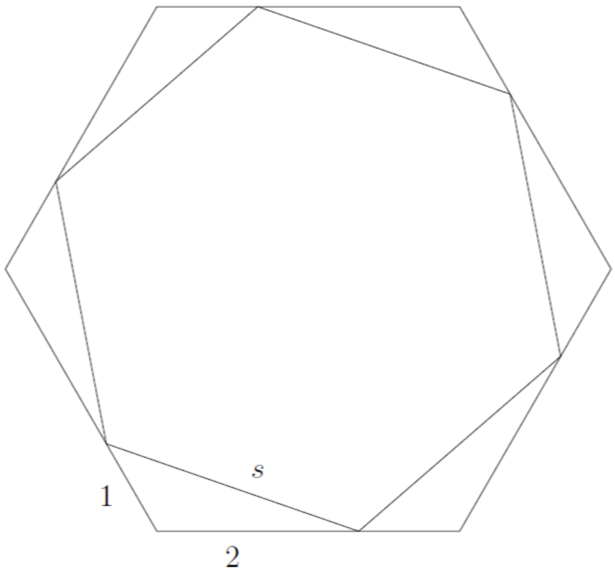
Answer: Mongolia

AVERAGE 45 seconds, 3 points

1. A regular hexagon is inscribed in another regular hexagon such that each vertex of the inscribed hexagon divides a side of the original hexagon into two parts in the ratio 2 : 1. Find the ratio of the area of the inscribed hexagon to the area of the larger hexagon.

   Answer: $\frac{7}{9}$

   Solution: Without loss of generality, we may assume that the original hexagon has side length 3. Let $s$ then be the side length of the inscribed hexagon. Notice that there are six triangles with sides 1, 2, and $s$, and the angle between the sides of lengths 1 and 2 is $120°$, as shown below:




   By the cosine law, we have

$$s^2 = 1^2 + 2^2 - 2 \cdot 1 \cdot 2 \cdot -\frac{1}{2} = 7$$

   and so $s = \sqrt{7}$. Since the ratio of the areas of two similar shapes is equal to the square of the ratio of their sides, the desired ratio is thus $\left(\frac{\sqrt{7}}{3}\right)^2 = \frac{7}{9}$.

2. The number $\sqrt{49 + 6\sqrt{6} + 12\sqrt{14} + 4\sqrt{21}}$ can be expressed as $a\sqrt{2} + b\sqrt{3} + c\sqrt{7}$ for some integers $a, b, c$. Find $a + b + c$.

   Answer: 6

   Solution: We seek for integers $a, b, c$ such that $\sqrt{49 + 6\sqrt{6} + 12\sqrt{14} + 4\sqrt{21}} = a\sqrt{2} + b\sqrt{3} + c\sqrt{7}$. The latter equation is equivalent to

$$\begin{aligned} 49 + 6\sqrt{6} + 12\sqrt{14} + 4\sqrt{21} &= (a\sqrt{2} + b\sqrt{3} + c\sqrt{7})^2 \\ &= 2a^2 + 3b^2 + 7c^2 + 2ab\sqrt{6} + 2ac\sqrt{14} + 2ca\sqrt{21}. \end{aligned}$$

   Comparing similar terms, we see that $(a, b, c)$ is the solution to the following system of equations

$$2a^2 + 3b^2 + 7c^2 \overset{(1)}{=} 49, \quad 2ab \overset{(2)}{=} 6, \quad 2ac \overset{(3)}{=} 12, \quad 2bc \overset{(4)}{=} 4.$$

   Multiplying equations $(2), (3), (4)$ yields $8(abc)^2 = 288$, so $abc = 6$. Hence, we get $c = 2$ from (2), $b = 1$ from (3) and $a = 3$ from (4). Hence, $a + b + c = 6$.

3. The letters of the word MATHEMATICS are rearranged to form distinct strings of the same 11 letters. What proportion of these strings do not contain the string MATH?

   Answer: $\frac{491}{495}$

Solution: By treating the string MATH as a single character, we find that there are exactly eight distinct "letters" to rearrange. Hence, there are 8! such words out of a total of $\frac{11!}{2!2!2!}$. This makes for a proportion of $\frac{8!\cdot 8}{11!} = \frac{4}{495}$, and so $\frac{491}{495}$ of the words do not contain the aforementioned string.

4. Determine $a$ and $b$ in the following:

$$(5!)^8 + (5!)^7 = 4a, 356, 487, b80, 000, 000$$

Answer: $a = 3, b = 6$

Solution: Observe that $(5!)^8 + (5!)^7 = (5!)^7 \times 121$. Hence it is divisible by 9 and 11.

Therefore,

$$4 + a + 3 + 5 + 6 + 4 + 8 + 7 + b + 8 = a + b + 45$$

is divisible by 9 and

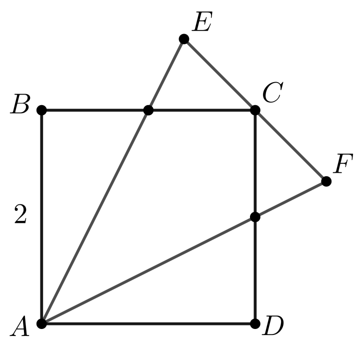
$$(4 + 3 + 6 + 8 + b) - (a + 5 + 4 + 7 + 8) = b - a - 3$$

is divisible by 11.

Hence $a + b = 0, 9, 18$ and $b - a = 3, 14, -8$. Since $a$ and $b$ are integers between 0 to 9, inclusive, then

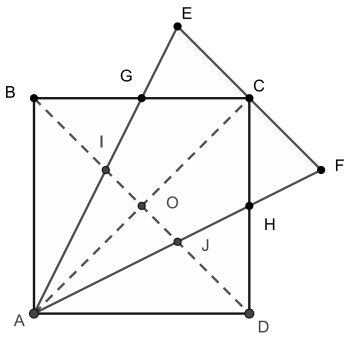
$$a + b = 9,\ b - a = 3 \Rightarrow a = 3, b = 6.$$

5. Given the square $ABCD$ with side length of 2, triangle $AEF$ is constructed so that $AE$ bisects side $BC$, and $AF$ bisects side $CD$. Moreover, $EF$ is parallel to the diagonal $BD$ and passes through $C$. Find the area of triangle $AEF$.




Answer: $\frac{8}{3}$

Solution: Refer to the figure below.




Observe that $\Delta GEC \cong \Delta BIG \cong \Delta FCH \cong \Delta HJD$.

Moreover, $BI = IJ = JD$. This is because $IO = \frac{1}{2}BI$ and $OJ = \frac{1}{2}JD$ (the points $I$ and $J$ are centroids)

Therefore:

$$\begin{aligned}[AEF] &= [AIJ] + [EIJF] \\ &= [AIJ] + [BCD] \\ &= \frac{1}{3}[BAD] + [BCD] \\ &= \frac{1}{3}(2) + 2 = \frac{8}{3}\end{aligned}$$

6. A semiprime is a number that is a product of two prime numbers. How many semiprime numbers less than 2018 can be expressed as $x^3 - 1$ for some natural number $x$?

   Answer: 4

   Solution: $x^3 - 1 = (x-1)(x^2+x+1)$, so this implies that both $x-1$ and $x^2+x+1$ have to be prime. Furthermore, this also means we only have numbers up to 12 to work on, as $13^3 > 2018$. Hence, we only have to check $x = 3, 4, 6, 8, 12$ and determine if $x^2 + x + 1$ is prime.

   The values we get are $13, 21, 43, 73, 157$, and of the 5, only 4 are prime.

7. Find the remainder when 53! is divided by 59.

   Answer: 30

   Solution: Let $x \equiv 53! \bmod 59$. By Wilson's theorem, $58! \equiv -1 \bmod 59$. Now, $58! = 58 \cdot 57 \cdot 56 \cdot 55 \cdot 54 \cdot 53! \equiv (-1)(-2)(-3)(-4)(-5)x \bmod 59$. This means that $-120x \equiv -1 \bmod 59$ or $2x \equiv 1 \bmod 59$. Thus, $x \equiv 60x \equiv 30 \bmod 59$.

8. Suppose that $\{a_n\}_{n\geq 1}$ is an increasing arithmetic sequence of integers such that $a_{a_{20}} = 17$ (where the subscript is $a_{20}$). Determine the value of $a_{2017}$.

   Answer: 4013

   Solution: Let $a_1 = a$ be the first term of such arithmetic sequence and $d > 0$ be its common difference. Then the condition $a_{a_{20}} = 17$ is equivalent to $17 = a + (a_{20} - 1)d = a + (a + 19d - 1)d = a(1+d) + 19d^2 - d$. Solving for $a$, we get

$$a = \frac{-19d^2 + d + 17}{d+1} = -19d + 20 - \frac{3}{d+1}.$$

   Since $a$ and $d$ are integers, $d + 1$ must divide 3. With $d > 0$, this forces $d + 1 = 3$ or $d = 2$, so $a = -19(2) + 20 - 1 = -19$. Hence, $a_{2017} = a + 2016d = -19 + 2016(2) = 4013$.

9. Let $N$ be the smallest positive integer such that $N/15$ is a perfect square, $N/10$ is a perfect cube, and $N/6$ is a perfect fifth power. Find the number of positive divisors of $N/30$.

   Answer: 8400

   Solution: $N$ must be of the form $N = 2^m 3^n 5^p$ for some nonnegative integers $m, n, p$. Since $N/15 = 2^m 3^{n-1} 5^{p-1}$ is a perfect square, we have $m \equiv 0 \bmod 2$ and $n \equiv p \equiv 1 \bmod 2$. Since $N/10 = 2^{m-1} 3^n 5^{p-1}$ is a perfect cube, we have $n \equiv 0 \bmod 3$ and $m \equiv p \equiv 1 \bmod 3$. Since $N/6 = 2^{m-1} 3^{n-1} 5^p$ is a perfect fifth power, we have $p \equiv 0 \bmod 5$ and $m \equiv n \equiv 1 \bmod 5$. By Chinese remainder theorem, we get $m \equiv 16 \bmod 30, n \equiv 21 \bmod 30$ and $p \equiv 25 \bmod 30$. Since $N$ is as small as possible, we take $m = 16, n = 21, p = 25$, so $N = 2^{16} 3^{21} 5^{25}$. Thus the number of positive divisors of $N/30 = 2^{15} 3^{20} 5^{24}$ is $16 \cdot 21 \cdot 25 = 8400$.

10. How many ordered quadruples $(a, b, c, d)$ of positive odd integers are there that satisfy the equation $a + b + c + 2d = 15$?

    Answer: 34

    Solution: Using the substitution $(a, b, c, d) = (2a_1 + 1, 2b_1 + 1, 2c_1 + 1, 2d_1 + 1)$ where $a_1, b_1, c_1, d_1$ are nonnegative integers, the problem is equivalent to finding the number $N$ of nonnegative integer solutions of $2a_1 + 1 + 2b_1 + 1 + 2c_1 + 1 + 4d_1 + 2 = 15$ or $a_1 + b_1 + c_1 + 2d_1 = 5$. Note that $d_1 \leq 2$, and for each fixed $d_1 \in \{0,1,2\}$, the number of nonnegative integer solutions of $a_1+b_1+c_1 = 5-2d_1$ is $\binom{7-2d_1}{2}$. Hence

$$N = \sum_{d_1=0}^{2} \binom{7-2d_1}{2} = \binom{7}{2} + \binom{5}{2} + \binom{3}{2} = 21+10+3 = 34.$$

DIFFICULT 90 seconds, 6 points

1. In triangle $ABC$, $AB = 6$, $BC = 10$, and $CA = 14$. If $D$, $E$, and $F$ are the midpoints of sides $BC$, $CA$, and $AB$, respectively, find $AD^2 + BE^2 + CF^2$.

   Answer: 249

   Solution: We use Stewart's Theorem with the median $AD$: (Note that $BD = CD = \frac{BC}{2}$.)

$$CA^2(BD) + AB^2(CD) = BC(AD^2 + (BD)(CD)) \to \frac{1}{2}\left(CA^2 + AB^2\right) = AD^2 + \left(\frac{BC}{2}\right)^2.$$

   Similarly, by using Stewart's Theorem with the medians $BE$ and $CF$, we get:

$$\frac{1}{2}\left(BC^2 + AB^2\right) = BE^2 + \left(\frac{CA}{2}\right)^2$$

   and

$$\frac{1}{2}\left(CA^2 + BC^2\right) = CF^2 + \left(\frac{AB}{2}\right)^2.$$

   Adding these three equations, we have:

$$AB^2 + BC^2 + CA^2 = AD^2 + BE^2 + CF^2 + \frac{1}{4}\left(AB^2 + BC^2 + CA^2\right).$$

   Thus,

$$AD^2 + BE^2 + CF^2 = \frac{3}{4}\left(AB^2 + BC^2 + CA^2\right) = \frac{3}{4}(6^2 + 10^2 + 14^2) = 249.$$

2. How many ways are there to arrange 5 identical red balls and 5 identical blue balls in a line if there cannot be three or more consecutive blue balls in the arrangement?

   Answer: 126

   Solution: We first consider the number of ways we can split the blue balls into groups of 1 or 2. The possible ways contain either 5 single blue balls, 3 single blue balls and one group of 2 balls, and 1 single blue ball with two groups of two balls. For each way, there are 1, 4, and 3 ways to arrange these groups.

   We can then add in the red balls as the dividers, meaning that there should be at least one red ball for to separate each group. The remaining red balls can be placed anywhere, using the Balls and Urns Method.

   - For the group of 5, you need 4 red balls to divide them, so there is one remaining ball and 6 urns, so the number of ways you can arrange the red ball is $\binom{1+6-1}{1} = 6$.
   - For the group of 3 single and 1 pair, you need 3 red balls, so the number of ways you can arrange the remaining red balls is $\binom{2+5-1}{2} = 15$.
   - For the group of 1 single and 2 pairs, you need 2 red balls, so the number of ways you can arrange the remaining red balls is $\binom{3+4-1}{3} = 20$.

   Thus, the total number of ways is $1*6 + 4*15 + 3*20 = 126$.

3. Let $x, y$, and $z$ be real numbers that satisfy the following system:

$$\begin{aligned} x + 3y + 6z &= 1 \\ xy + 2xz + 6yz &= -8 \\ xyz &= 2 \end{aligned}$$

Find the smallest possible value of $x + y + z$.

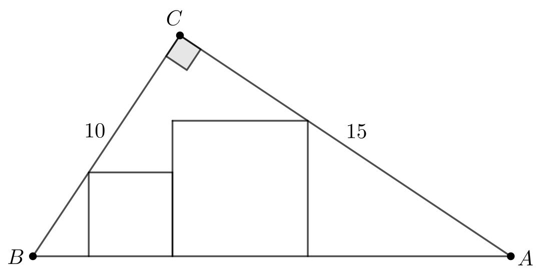
Answer: $-\dfrac{8}{3}$

Solution: Let $y' = 3y$, $z' = 6z$. The system of equations can then be rewritten as

$$\begin{aligned} x + y' + z' &= 1 \\ xy' + xz' + y'z' &= -24 \\ xy'z' &= 36 \end{aligned}$$

and so $x, y', z'$ are the roots of the polynomial $t^3 - t^2 - 24t - 36 = 0$. This factors into $(t-6)(t+3)(t+2) = 0$, and so, in some order, $x, y', z'$ must be $6, -3, -2$. This gives us the following solutions: $(6, -1, -\frac{1}{3}), (6, -\frac{2}{3}, -\frac{1}{2}), (-3, 2, -\frac{1}{3}), (-3, -\frac{2}{3}, 1), (-2, -1, 1), (-2, 2, -\frac{1}{2})$. Among these, the ordered triple which gives the minimum sum is $(-3, -\frac{2}{3}, 1)$, which has a sum of $-8/3$.

4. In a right triangle $ABC$ with $\angle C = 90°$, $BC = 10$, and $AC = 15$. Two squares are inscribed in $ABC$ as shown in the figure. Find the minimum sum of the areas of the squares.




Answer: 36

Solution: Let $NM = s$ and $SR = t$ be the side lengths of the two squares. By the Pythagorean theorem, we have $AB = 5\sqrt{13}$. From the above figure, triangles $NMB, ACB$ and $ARS$ are similar. Thus, $MB = \frac{BC \cdot NM}{AC} = \frac{2s}{3}$ and $AR = \frac{AC \cdot RS}{BC} = \frac{3t}{2}$. We see that $5\sqrt{13} = AB = AR + RQ + QM + MB = \frac{5s}{3} + \frac{5t}{2}$ or $\sqrt{13} = \frac{s}{3} + \frac{t}{2}$. Hence, by Cauchy-Schwarz inequality, we get

$$(s^2 + t^2)\left(\frac{1}{3^2} + \frac{1}{2^2}\right) \geq \left(\frac{s}{3} + \frac{t}{2}\right)^2 = 13$$

so that $s^2 + t^2 \geq 13/(\frac{1}{9} + \frac{1}{4}) = 36$. Hence, the minimum sum of the areas of the squares $S_1$ and $S_2$ is 36, which is attained when $s = 12/\sqrt{13}$ and $t = 18/\sqrt{13}$.

5. Let $P(x)$ be the polynomial of minimal degree such that $P(k) = 720k/(k^2 - 1)$ for $k \in \{2, 3, 4, 5\}$. Find the value of $P(6)$.

Answer: 48

Solution: Let $Q(x) = (x^2 - 1)P(x) - 720x$. Then $Q(k) = 0$ for $k \in \{2, 3, 4, 5\}$ so $Q(x) = R(x)(x-2)(x-3)(x-4)(x-5)$ for some polynomial $R(x)$. Observe that $Q(1) = -720$ and $Q(-1) = 720$, so $R(x)$ cannot be constant. As the degree of $Q$ is as small as possible, we set $R(x) = ax + b$ for some constants $a$ and $b$ so that $Q(x) \overset{(\star)}{=} (ax + b)(x-2)(x-3)(x-4)(x-5)$. Letting $x = 1$ on $(\star)$, we have $a + b \overset{(1)}{=} -30$ and letting $x = -1$ on $(\star)$, we have $-a + b = 2$. From (1) and (2), we get $(a, b) = (-16, -14)$. Thus,

$$(x^2 - 1)P(x) - 720x = Q(x) = (-16x - 14)(x-2)(x-3)(x-4)(x-5)$$

and letting $x = 6$ yields $35P(6) - 720(6) = -110(24) = -2640$. Hence, $P(6) = 48$.

SPARE

1. (Easy) Find the largest value of $x$ such that $\sqrt[3]{x}+\sqrt[3]{10-x}=1$.

   Answer: $5+2\sqrt{13}$

   Solution: Cubing both sides of the given equation yields $x+3\sqrt[3]{x(10-x)}(\sqrt[3]{x}+\sqrt[3]{10-x})+10-x=1$, which then becomes $10+3\sqrt[3]{x(10-x)}=1$ or $\sqrt[3]{x(10-x)}\overset{(1)}{=}-3$. Cubing both sides of equation (1) gives $x(10-x)=-27$ or $x^2-10x=27$. This means $(x-5)^2=52$ so $x=5\pm 2\sqrt{13}$ and the largest real solution is $x=5+2\sqrt{13}$.

2. (Easy/Average) Find the remainder when $14^{100}$ is divided by 45.

   Answer: 31

   Solution: Let $\Phi$ denote the Euler's totient function. Since the prime factors of 45 are 3 and 5 only, then
   $$\Phi(45)=45\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)=24.$$
   Since $gcd(14,45)=1$, by Euler's Theorem, $14^{24}\equiv 1(\text{mod }45)$. So that
   $$14^{100}=(14^{24})^4(14^4)\equiv 14^4(\text{mod }45)\equiv 196^2(\text{mod }45)\equiv 16^2(\text{mod }45)\equiv 31(\text{mod }45).$$

3. (Easy/Average) Given that $\tan x+\cot x=8$, find the value of $\sqrt{\sec^2 x+\csc^2 x-\frac{1}{2}\sec x\csc x}$.

   Answer: $2\sqrt{15}$

   Solution: Note that
   $$\tan x+\cot x=\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}=\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}=\frac{1}{\sin x\cos x}.$$
   This means that $\sin x\cos x=\dfrac{1}{8}$.

   Now,
   $$\begin{aligned}\sqrt{\sec^2 x+\csc^2 x-\frac{1}{2}\sec x\csc x} &= \sqrt{\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}-\frac{1}{2}\left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right)}\\ &= \sqrt{\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}-\left(\frac{1}{2}\right)(8)}\\ &= =\sqrt{64-4}=2\sqrt{15}.\end{aligned}$$

4. (Average) Simplify $\sqrt[3]{5\sqrt{2}+7}-\sqrt[3]{5\sqrt{2}-7}$ into a rational number.

   Answer: 2

   Solution: Let $a=\sqrt[3]{5\sqrt{2}+7}$ and let $b=\sqrt[3]{5\sqrt{2}-7}$. Note that
   $$a^3-b^3=14 \text{ and } ab=\sqrt[3]{50-49}=1.$$
   Now, $a^3-b^3=(a-b)(a^2+ab+b^2)=(a-b)[(a-b)^2+3ab]$. Letting $x=a-b$, we have the resulting equation $x(x^2+3)=14$.
   $$x(x^2+3)=14\rightarrow x^3+3x-14=0\rightarrow (x-2)(x^2+2x+7)=0.$$
   The roots of $x^2+2x+7$ are not real since its discriminant is $2^2-4(7)=-24<0$. Since $x=a-b$ is a real number, then $x=2$.

   (Alternatively, the expression is equivalent to $(\sqrt{2}+1)-(\sqrt{2}-1)=2$.)

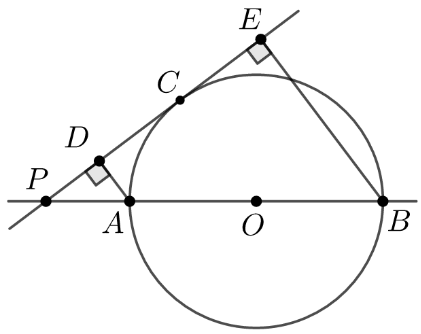
5. (Average) The sum of the terms of an infinite geometric series is 2 and the sum of the squares of the corresponding terms of this series is 6. Find the sum of the cubes of the corresponding terms.

Answer: $\dfrac{96}{7}$

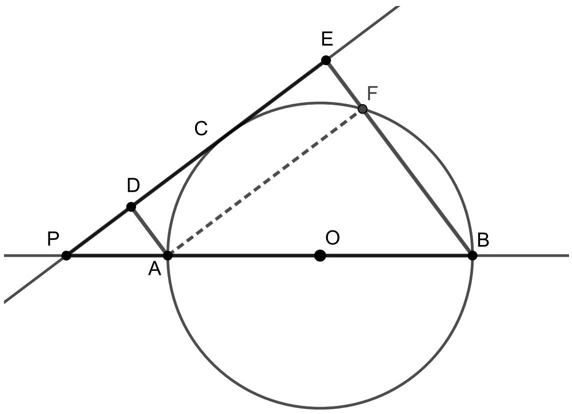
Solution: Let $a$ be the first term and let $r \in (-1, 1)$ be the common ratio of such infinite geometric series. Then, $a/(1-r) \overset{(1)}{=} 2$ and $a^2/(1-r^2) \overset{(2)}{=} 6$. Squaring (1) gives $a^2/(1-r)^2 = 4$ and using (2) yields $(1-r)/(1+r) = 3/2$. Solving for $r$, we get $r = -\frac{1}{5}$ so from (1) we get $a = 2(1-r) = \frac{12}{5}$. Thus,

$$\frac{a^3}{1-r^3} = \frac{\frac{12^3}{5^3}}{1+\frac{1}{5^3}} = \frac{12^3}{5^3+1} = \frac{12 \cdot 12^2}{6(25-5+1)} = \frac{288}{21} = \frac{96}{7}.$$

6. (Difficult) On the line containing diameter AB of a circle, a point P is chosen outside of this circle, with P closer to A than B. One of the two tangent lines through P is drawn. Let $D$ and $E$ be two points on the tangent line such that $AD$ and $BE$ are perpendicular to it. If $DE = 6$, find the area of triangle $BEP$.




Answer: $12 + \frac{69}{14}\sqrt{7}$




Solution:

$FB = \sqrt{AB^2 - AF^2} = \sqrt{64-36} = 2\sqrt{7}$

The radius $OC$ is the midsegment of trapezoid $ADEB$. Hence $[ABED] = 4 \times 6 = 24$. Moreover

$$\begin{aligned}
[ABED] &= [ABF] + [ADEF] \\
24 &= \frac{2\sqrt{7} \times 6}{2} + AD \times 6 \\
AD &= 4 - \sqrt{7} \\
EB &= AD + FB = 4 + \sqrt{7}
\end{aligned}$$

Also by similarity (triangle $PAD$ and triangle $PBE$)

$$\begin{aligned}
\frac{PD}{AD} &= \frac{PE}{EB} \\
\frac{PD}{4-\sqrt{7}} &= \frac{PD+6}{4+\sqrt{7}} \\
4PD + \sqrt{7}PD &= 4PD - \sqrt{7}PD + 24 - 6\sqrt{7} \\
2\sqrt{7}PD &= 24 - 6\sqrt{7} \\
PD &= \frac{12}{\sqrt{7}} - 3 = \frac{12\sqrt{7}}{7} - 3
\end{aligned}$$

Hence the area is equal to

$$\frac{1}{2}\left(\frac{12\sqrt{7}}{7}-3\right)\left(4+\sqrt{7}\right)=12+\frac{69}{14}\sqrt{7}$$