20th Philippine Mathematical Olympiad

Mathematical Society of the Philippines

Department of Science and Technology - Science Education Institute

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National Stage, Written Phase

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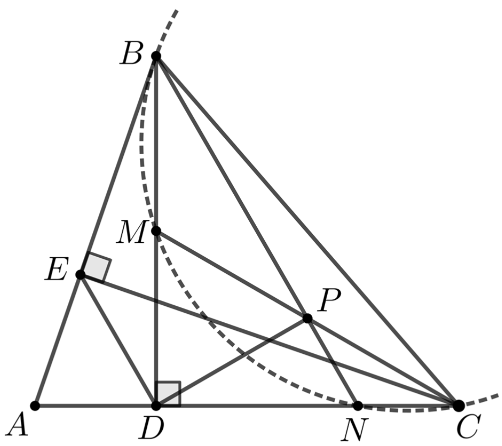
Time: 4.5 hours *Each item is worth 7 points.*

1. In triangle ABC with $\angle ABC = 60^\circ$ and $5AB = 4BC$, points D and E are the feet of the altitudes from B and C, respectively. M is the midpoint of BD and the circumcircle of triangle BMC meets line AC again at N. Lines BN and CM meet at P. Prove that $\angle EDP = 90^\circ$.

<u>Solution:</u> From the given, $AB = 4l$ and $BC = 5l$ for some constant $l > 0$. Since $\angle ABC = 60^\circ$, $BE = \frac{5l}{2}$ and $CE = \frac{5\sqrt{3}l}{2}$. Also, by the cosine law, $AC = \sqrt{21}l$. Since $BEDC$ is cyclic, $\angle EDA = \angle ABC = 60^\circ$. Consequently, $\angle EDB = 30^\circ$ and $\triangle AED \sim \triangle ACB$. From the latter, $AD = 4k$, $DE = 5k$, and $AE = \sqrt{21}k$ for some constant $k > 0$. Since $4l = AB = BE + AE = \frac{5l}{2} + \sqrt{21}k$, then $\frac{l}{k} = \frac{2\sqrt{21}}{3}$.



The area of $\triangle ABC$ equals

$$\frac{1}{2}\sin 60^\circ \cdot 4l \cdot 5l = \frac{1}{2} \cdot \sqrt{21}l \cdot 2BM$$

which gives $BM = \frac{5l}{\sqrt{7}}$. Observe that

$$\frac{CE}{DE} = \frac{5\sqrt{3}l/2}{5k} = \frac{\sqrt{3}l}{2k} = \frac{\sqrt{3}}{2} \cdot \frac{2\sqrt{21}}{3} = \sqrt{7} = \frac{5l}{5l/\sqrt{7}} = \frac{CB}{MB}.$$

This, along with $\angle MBC = \angle DBC = \angle DEC$, implies that $\triangle DEC \sim \triangle MBC$, so $\angle ECD = \angle BCM$ and thus, $\angle MCD = \angle BCE = 30^\circ$. As $BMNC$ is cyclic, $\angle MBN = 30^\circ$ so that lines ED and BN are parallel. We have $\angle DMC = 60^\circ$ so that $\angle BPM = 30^\circ$. Thus, $\triangle BMP$ is isosceles with $BM = MP$ and it follows that M is the circumcenter of $\triangle BPD$. Therefore, $\angle BPD = 90^\circ$. It follows that $\angle EDP = 90^\circ$. $\square$

2. Suppose $a_1, a_2, \ldots$ is a sequence of integers, and d is some integer. For all natural numbers n,

(i) $|a_n|$ is prime; (ii) $a_{n+2} = a_{n+1} + a_n + d$.

Show that the sequence is constant.

<u>*Solution:*</u> Consider the sequence $\{b_n\}$ defined by $b_n = a_n + d$ for all n, so that $b_{n+2} = b_{n+1} + b_n$ for all n. This sequence is determined by its first two terms b_1 and b_2, and the same holds true if we reduce the sequence mod a_1. Taking remainders mod a_1, pairs of consecutive terms will repeat themselves, and so the sequence $\{b_n\}$ is periodic mod a_1, i.e., there exists a positive integer ℓ for which $b_{k+\ell} \equiv b_\ell \pmod{a_1}$ for all k, and so $a_{\ell+1} \equiv a_1 \pmod{a_1}$. Thus, $a_1 \mid a_{\ell+1}$. From (i), we must have $|a_1| = |a_{\ell+1}|$, and in fact, $|a_1| = |a_{k\ell+1}|$ for all k. In particular, $a_{k\ell+1}$, and thus $b_{k\ell+1}$, assumes at most two distinct values.

It suffices to show that $\{b_n\}$ is constant. Consider the characteristic polynomial of the recurrence defining $\{b_n\}$, $P(x) = x^2 - x - 1$. Let φ and ψ be the distinct roots of P, with $\varphi > \psi$. Note that in fact $\varphi > 1$ while $0 > \psi > -1$. There exists a unique pair of constants c_1, c_2, dependent on the values of b_1 and b_2, satisfying the system

$$\begin{aligned} b_1 &= c_1 + c_2 \\ b_2 &= c_1\varphi + c_2\psi. \end{aligned}$$

It can be proved easily by induction that $b_n = c_1\varphi^{n-1} + c_2\psi^{n-1}$ for all $n \geq 1$. From this, we get that $|b_n| \geq |c_1|\varphi^{n-1} - |c_2|$. If $c_1 \neq 0$, then $|b_n|$ eventually grows without bound, which contradicts our previous assertion that $|b_{k\ell+1}|$ assumes at most two values. Thus, $c_1 = 0$, and consequently, $b_{n+1} = \psi b_n$. However, b_{n+1} and b_n are integers while ψ is irrational. This forces us to conclude that $b_{n+1} = b_n = 0$, and so $c_2 = 0$ as well. Thus, $b_n = 0$ for all n, and $a_n = a_1$ (with $d = -a_1$). □

3. Let n be a positive integer. An $n \times n$ matrix (a rectangular array of numbers with n rows and n columns) is said to be a platinum[1] matrix if

(i) the n^2 entries are integers from 1 to n;

(ii) each row, each column, and the main diagonal (from the upper left corner to the lower right corner) contains each integer from 1 to n exactly once; and

[1]Platinum is the modern gift for the 20th wedding anniversary.

(iii) there exists a collection of n entries containing each of the numbers from 1 to n, such that no two entries lie on the same row or column, and none of which lie on the main diagonal of the matrix.

Determine all values of n for which there exists an $n \times n$ platinum matrix.

This is based on the paper *A Simple Method for Constructing Doubly Diagonalized Latin Squares* by Ervin Gergely, Journal of Combinatorial Theory (A) 16, 266-272 (1974).

<u>*Solution:*</u> There is no platinum matrix for $n = 1$ and for $n = 2$. We claim that a platinum matrix exists for all integers $n \geq 3$.

Define a *transversal* as a collection of n matrix entries which are taken from distinct rows and columns, and which contains each of the numbers 1 to n.

When $n \geq 3$ is odd, we can construct an $n \times n$ platinum matrix in the following manner: First, fill up the first row with $1, n, n-1, \ldots, 3, 2$ from left to right. Then fill up each "diagonal" (which goes up to down, left to right and wraps back to the 1st column after the nth column) by starting with its first row entry and incrementing by 1 as we go down to the last row. From this point onwards, entries are reduced mod n, but with n written instead of 0. For example, for $n = 3$, we have the following platinum matrix.

$$\begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} + \begin{pmatrix} & 3 & \\ & & 1 \\ 2 & & \end{pmatrix} + \begin{pmatrix} & & 2 \\ 3 & & \\ & 1 & \end{pmatrix}$$

From the construction described, the entry in row i, column j ($1 \leq i, j \leq n$) is $a_{ij} = 2i - j$ (again, reduced mod n as mentioned above). For a fixed row i, the entries for different columns j and j' are distinct. For a fixed column j, the entries for different rows i and i' are distinct since n is odd. The diagonal whose first row entry is 2 is a transversal; in fact, each diagonal is a transversal. Thus, the matrix is platinum.

Let $n \geq 8$ be even, so $n - 3 \geq 5$ is odd. Consider the $(n-3) \times (n-3)$ platinum matrix, denoted by C_{n-3}, following the construction above. We start constructing our $n \times n$ platinum matrix as follows:

$$\left(\begin{array}{c|c} C_{n-3} & \\ \hline & C_3 \end{array}\right) \qquad \text{where } C_3 = \begin{pmatrix} n-2 & n & n-1 \\ n & n-1 & n-2 \\ n-1 & n-2 & n \end{pmatrix}.$$

We then need to fill in the $3 \times (n-3)$ matrix and the $(n-3) \times 3$ matrix adjacent to C_{n-3}. To do this, from the $(n-3) - 1 = n - 4 \geq 4$ diagonals of

C_{n-3} other than its main diagonal, choose 3. For one of these transversals, project its entries vertically into an empty row and horizontally into an empty column, then replace all of the entries of this chosen transversal by $n-2$. Then repeat this procedure using the symbols $n-1$ and n using the two other transversals. The resulting matrix is then a platinum matrix. Of the $n-4 \geq 4$ transversals described above for C_{n-3}, at least one has not been used yet. This, along with a "transversal" of C_3 other than its main diagonal, then form a transversal of the formed $n \times n$ matrix; none of the entries of this transversal are in the main diagonal. The matrix formed is thus platinum.

The following illustrates the construction for the case $n = 8$. The final matrix is platinum, with the boxed entries all off-diagonal and forming a transversal.

$$\left(\begin{array}{ccccc|ccc}
1 & 5 & 4 & 3 & 2 & & & \\
3 & 2 & 1 & 5 & 4 & & & \\
5 & 4 & 3 & 2 & 1 & & & \\
2 & 1 & 5 & 4 & 3 & & & \\
4 & 3 & 2 & 1 & 5 & & & \\
\hline
 & & & & & 6 & 8 & 7 \\
 & & & & & 8 & 7 & 6 \\
 & & & & & 7 & 6 & 8
\end{array}\right) \longrightarrow \left(\begin{array}{ccccc|ccc}
1 & 6 & 4 & 3 & 2 & 5 & & \\
3 & 2 & 6 & 5 & 4 & 1 & & \\
5 & 4 & 3 & 6 & 1 & 2 & & \\
2 & 1 & 5 & 4 & 6 & 3 & & \\
6 & 3 & 2 & 1 & 5 & 4 & & \\
\hline
4 & 5 & 1 & 2 & 3 & 6 & 8 & 7 \\
 & & & & & 8 & 7 & 6 \\
 & & & & & 7 & 6 & 8
\end{array}\right) \longrightarrow$$

$$\left(\begin{array}{ccccc|ccc}
1 & 6 & 7 & 3 & 2 & 5 & 4 & \\
3 & 2 & 6 & 7 & 4 & 1 & 5 & \\
5 & 4 & 3 & 6 & 7 & 2 & 1 & \\
7 & 1 & 5 & 4 & 6 & 3 & 2 & \\
6 & 7 & 2 & 1 & 5 & 4 & 3 & \\
\hline
4 & 5 & 1 & 2 & 3 & 6 & 8 & 7 \\
2 & 3 & 4 & 5 & 1 & 8 & 7 & 6 \\
 & & & & & 7 & 6 & 8
\end{array}\right) \longrightarrow \left(\begin{array}{ccccc|ccc}
1 & 6 & 7 & 8 & \boxed{2} & 5 & 4 & 3 \\
\boxed{3} & 2 & 6 & 7 & 8 & 1 & 5 & 4 \\
8 & \boxed{4} & 3 & 6 & 7 & 2 & 1 & 5 \\
7 & 8 & \boxed{5} & 4 & 6 & 3 & 2 & 1 \\
6 & 7 & 8 & \boxed{1} & 5 & 4 & 3 & 2 \\
\hline
4 & 5 & 1 & 2 & 3 & 6 & \boxed{8} & 7 \\
2 & 3 & 4 & 5 & 1 & 8 & 7 & \boxed{6} \\
5 & 1 & 2 & 3 & 4 & \boxed{7} & 6 & 8
\end{array}\right)$$

Lastly, consider the following matrices.

$$\begin{pmatrix}
1 & 2 & 3 & 4 \\
3 & 4 & 1 & 2 \\
4 & 3 & 2 & 1 \\
2 & 1 & 4 & 3
\end{pmatrix} \qquad \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
5 & 3 & 6 & 1 & 4 & 2 \\
4 & 1 & 5 & 2 & 6 & 3 \\
2 & 4 & 1 & 6 & 3 & 5 \\
6 & 5 & 4 & 3 & 2 & 1 \\
3 & 6 & 2 & 5 & 1 & 4
\end{pmatrix}$$

It is straightforward to verify that these are platinum matrices for $n = 4$ and $n = 6$. $\square$

4. Determine all ordered pairs (x, y) of nonnegative integers that satisfy the equation

$$3x^2 + 2 \cdot 9^y = x(4^{y+1} - 1).$$

This is a modified version of Problem N2 from the 2010 IMO shortlist.

<u>*Solution:*</u> The equation is equivalent to

$$(3x)^2 + 2 \cdot 3^{2y+1} = 3x \left[2^{(2y+1)+1} - 1 \right].$$

Letting $a = 3x$ and $b = 2y + 1$, we have

$$a^2 + 2 \cdot 3^b = a(2^{b+1} - 1). \tag{1}$$

<u>**Case 1:**</u> $b = 1$

We have $a^2 - 3a + 6 = 0$ which has no integer solution for a. Thus, there is no solution in this case.

<u>**Case 2:**</u> $b = 3$

We have $a^2 - 15a + 54 = 0$, whose roots are 6 and 9, both divisible by 3. This case then has the solutions $(2, 1)$ and $(3, 1)$ for the original equation.

<u>**Case 3:**</u> $b = 5$

We have $a^2 - 63a + 486 = 0$, whose roots are 9 and 54, both divisible by 3. We get as additional solutions $(3, 2)$ and $(18, 2)$.

<u>**Case 4:**</u> $b \geq 7$, b odd

It follows from (1) that $a | 2 \cdot 3^b$. Since a is divisible by 3, either $a = 3^p$ for some $1 \leq p \leq b$ or $a = 2 \cdot 3^q$ for some $1 \leq q \leq b$.

For the first case $a = 3^p$, let $q = b - p$. Then we have

$$2^{b+1} - 1 = a + \frac{2 \cdot 3^b}{a} = 3^p + 2 \cdot 3^q.$$

For the second case $a = 2 \cdot 3^q$, let $p = b - q$. Then we have

$$2^{b+1} - 1 = a + \frac{2 \cdot 3^b}{a} = 2 \cdot 3^q + 3^p.$$

In either case,

$$2^{b+1} - 1 = 3^p + 2 \cdot 3^q \tag{2}$$

where $p + q = b$. Consequently, $2^{b+1} > 3^p$ and $2^{b+1} > 2 \cdot 3^q$. Thus,

$$3^p < 2^{b+1} = 8^{\frac{b+1}{3}} < 9^{\frac{b+1}{3}} = 3^{\frac{2(b+1)}{3}}$$

$$2 \cdot 3^q < 2^{b+1} = 2 \cdot 8^{\frac{b}{3}} < 2 \cdot 9^{\frac{b}{3}} = 2 \cdot 3^{\frac{2b}{3}} < 2 \cdot 3^{\frac{2(b+1)}{3}}$$

Thus, $p, q < \frac{2(b+1)}{3}$. Since $p = b-q$ and $q = b-p$, we get $p, q > b-\frac{2(b+1)}{3} = \frac{b-2}{3}$. Therefore,

$$\frac{b-2}{3} < p, q < \frac{2(b+1)}{3}.$$

Let $r = \min\{p, q\}$. Since $r > \frac{b-2}{3} \geq \frac{5}{3}$, then $r \geq 2$. Consequently, the right hand side of (2) is divisible by 9. Thus, 9 divides $2^{b+1} - 1$. This is true only if $6|b+1$. Since $b \geq 7$, then $b \geq 11$. Thus, we can let $b+1 = 6s$ for some positive integer s, and we can write

$$2^{b+1} - 1 = 2^{6s} - 1 = 4^{3s} - 1 = (2^s - 1)(2^s + 1)(4^{2s} + 4^s + 1).$$

Since $4^s \equiv 1 \mod 3$, then $4^{2s}+4^s+1 = (4^s-1)^2+3\cdot 4^s$ is always divisible by 3 but never by 9. Furthermore, at most one of $2^s - 1$ and $2^s + 1$ is divisible by 3, being consecutive odd numbers. Since $3^r|2^{b+1} - 1$, then either $3^{r-1}|2^s - 1$ or $3^{r-1}|2^s + 1$. From both cases, we have $3^{r-1} \leq 2^s + 1$. Thus,

$$3^{r-1} \leq 2^s + 1 \leq 3^s = 3^{\frac{b+1}{6}}.$$

Therefore,

$$\frac{b-2}{3} - 1 < r - 1 \leq \frac{b+1}{6}$$

which implies $b < 11$. However, there is no odd integer b between 7 (included) and 11 (excluded) such that $6|b+1$. There are then no solutions for this last case.

Therefore, the solutions (x, y) of the given equation are $(2, 1)$, $(3, 1)$, $(3, 2)$, and $(18, 2)$. □

The PHILIPPINE TEAM to the
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