20th Philippine Mathematical Olympiad

Qualifying Stage, 28 October 2017

A project of the **Mathematical Society of the Philippines (MSP)** and the **Department of Science and Technology - Science Education Institute (DOST-SEI)** in partnership with **HARI Foundation** and **Manulife Business Processing Services**

PART I. Choose the best answer. Each correct answer is worth two points.

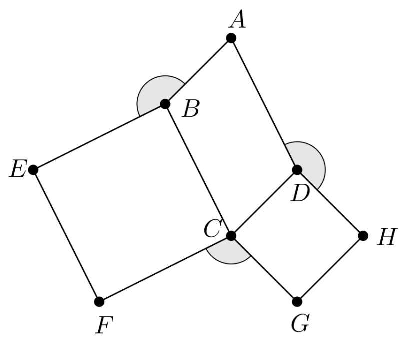
1. Find x if $\frac{79}{125}\left(\frac{79+x}{125+x}\right)=1$.

   (a) 0 (b) -46 (c) -200 (d) -204

2. The line $2x+ay=5$ passes through $(-2,-1)$ and $(1,b)$. What is the value of b?

   (a) $-\frac{1}{2}$ (b) $-\frac{1}{3}$ (c) $-\frac{1}{4}$ (d) $-\frac{1}{6}$

3. Let $ABCD$ be a parallelogram. Two squares are constructed from its adjacent sides, as shown in the figure below. If $\angle BAD = 56°$, find $\angle ABE + \angle ADH + \angle FCG$, the sum of the three highlighted angles.



   (a) 348° (b) 384° (c) 416° (d) 432°

4. For how many integers x from 1 to 60, inclusive, is the fraction $\frac{x}{60}$ already in lowest terms?

   (a) 15 (b) 16 (c) 17 (d) 18

5. Let r and s be the roots of the polynomial $3x^2-4x+2$. Which of the following is a polynomial with roots $\frac{r}{s}$ and $\frac{s}{r}$?

   (a) $3x^2+2x+3$ (b) $3x^2+2x-3$ (c) $3x^2-2x+3$ (d) $3x^2-2x-3$

6. If the difference between two numbers is a and the difference between their squares is b, where $a, b > 0$, what is the sum of their squares?

(a) $\dfrac{a^2+b^2}{a}$ (b) $2\left(\dfrac{a+b}{a}\right)^2$ (c) $\left(a+\dfrac{b}{a}\right)^2$ (d) $\dfrac{a^4+b^2}{2a^2}$

7. Evaluate the sum

$$\sum_{n=3}^{2017} \sin\left(\frac{(n!)\pi}{36}\right).$$

(a) 0 (b) $\dfrac{1}{2}$ (c) $-\dfrac{1}{2}$ (d) 1

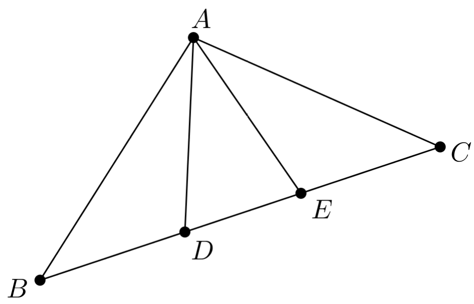
8. In $\triangle ABC$, D is the midpoint of BC. If the sides AB, BC, and CA have lengths $4, 8,$ and 6, respectively, then what is the numerical value of AD^2 ?

(a) 8 (b) 10 (c) 12 (d) 13

9. Let A be a positive integer whose leftmost digit is 5 and let B be the number formed by reversing the digits of A. If A is divisible by 11, 15, 21, and 45, then B is <u>not always</u> divisible by

(a) 11 (b) 15 (c) 21 (d) 45

10. In $\triangle ABC$, the segments AD and AE trisect $\angle BAC$. Moreover, it is also known that $AB = 6, AD = 3, AE = 2.7, AC = 3.8$ and $DE = 1.8$. The length of BC is closest to which of the following?



(a) 8 (b) 8.2 (c) 8.4 (d) 8.6

11. Let $\{a_n\}$ be a sequence of real numbers defined by the recursion $a_{n+2} = a_{n+1} - a_n$ for all positive integers n. If $a_{2013} = 2015$, find the value of $a_{2017} - a_{2019} + a_{2021}$.

(a) 2015 (b) -2015 (c) 4030 (d) -4030

12. A *lattice point* is a point whose coordinates are integers. How many lattice points are strictly inside the triangle formed by the points $(0,0)$, $(0,7)$, and $(8,0)$?

(a) 21 (b) 22 (c) 24 (d) 28

13. Find the sum of the solutions to the logarithmic equation

$$x^{\log x} = 10^{2-3\log x+2(\log x)^2},$$

where $\log x$ is the logarithm of x to the base 10.

(a) 10 (b) 100 (c) 110 (d) 111

14. Triangle ABC has $AB = 10$ and $AC = 14$. A point P is randomly chosen in the interior or on the boundary of triangle ABC. What is probability that P is closer to AB than to AC?

(a) 1/4 (b) 1/3 (c) 5/7 (d) 5/12

15. Suppose that $\{a_n\}$ is a nonconstant arithmetic sequence such that $a_1 = 1$ and the terms a_3, a_{15}, a_{24} form a geometric sequence in that order. Find the smallest index n for which $a_n < 0$.

(a) 50 (b) 51 (c) 52 (d) 53

PART II. Choose the best answer. Each correct answer is worth three points.

1. Two red balls, two blue balls, and two green balls are lined up into a single row. How many ways can you arrange these balls such that no two adjacent balls are of the same color?

(a) 15 (b) 30 (c) 60 (d) 90

2. What is the sum of the last two digits of $403^{(10^{10}+6)}$?

(a) 9 (b) 10 (c) 11 (d) 12

3. How many strictly increasing finite sequences (having one or more terms) of positive integers less than or equal to 2017 with an odd number of terms are there?

(a) 2^{2016} (b) $\dfrac{4034!}{(2017!)^2}$ (c) $2^{2017} - 2017^2$ (d) $2^{2018} - 1$

4. If one of the legs of a right triangle has length 17 and the lengths of the other two sides are integers, then what is the radius of the circle inscribed in that triangle?

(a) 8 (b) 14 (c) 11 (d) 10

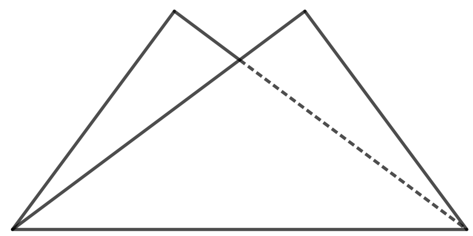
5. Let N be the smallest three-digit positive number with exactly 8 positive even divisors. What is the sum of the digits of N?

(a) 4 (b) 9 (c) 12 (d) 13

6. Let a, b, c be randomly chosen (in order, and with replacement) from the set $\{1, 2, 3, \ldots, 999\}$. If each choice is equally likely, what is the probability that $a^2 + bc$ is divisible by 3?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{7}{27}$ (d) $\frac{8}{27}$

7. Folding a rectangular sheet of paper with length ℓ and width w in half along one of its diagonals, as shown in the figure below, reduces its "visible" area (the area of the pentagon below) by 30%. What is $\frac{\ell}{w}$?



(a) $\frac{4}{3}$ (b) $\frac{2}{\sqrt{3}}$ (c) $\sqrt{5}$ (d) $\frac{\sqrt{5}}{2}$

8. Find the sum of all positive integers k such that $k(k + 15)$ is a perfect square.

(a) 63 (b) 65 (c) 67 (d) 69

9. Let $f(n) = \frac{n}{3^r}$ where n is an integer, and r is the largest nonnegative integer such that n is divisible by 3^r. Find the number of distinct values of $f(n)$ where $1 \leq n \leq 2017$.

(a) 1344 (b) 1345 (c) 1346 (d) 1347

10. If A, B, and C are the angles of a triangle such that

$$5 \sin A + 12 \cos B = 15$$

and

$$12 \sin B + 5 \cos A = 2,$$

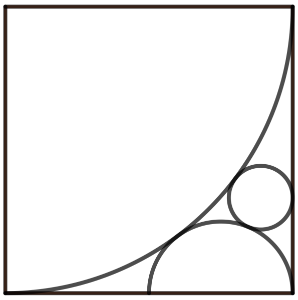
then the measure of angle C is

(a) 150° (b) 135° (c) 45° (d) 30°

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. How many three-digit numbers are there such that the sum of two of its digits is the largest digit?

2. In the figure, a quarter circle, a semicircle and a circle are mutually tangent inside a square of side length 2. Find the radius of the circle.



3. Find the minimum value of
$$\frac{18}{a+b} + \frac{12}{ab} + 8a + 5b,$$
where a and b are positive real numbers.

4. Suppose $\dfrac{\tan x}{\tan y} = \dfrac{1}{3}$ and $\dfrac{\sin 2x}{\sin 2y} = \dfrac{3}{4}$, where $0 < x, y < \dfrac{\pi}{2}$. What is the value of $\dfrac{\tan 2x}{\tan 2y}$?

5. Find the largest positive real number x such that
$$\frac{2}{x} = \frac{1}{\lfloor x \rfloor} + \frac{1}{\lfloor 2x \rfloor},$$
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Answers

Part I. (2 points each)

1. D
2. B
3. C
4. B
5. C
6. D
7. B
8. B
9. C
10. A
11. D
12. A
13. C
14. D
15. C

Part II. (3 points each)

1. B
2. C
3. A
4. A
5. B
6. A
7. C
8. C
9. B
10. D

Part III. (6 points each)

1. 279 (or 126)[1]
2. $\frac{2}{9}$
3. 30
4. $-\frac{3}{11}$
5. $\frac{20}{7}$

[1] We are also accepting the answer 126, as the wording of the problem seems to suggest that the sum of the two digits is equal to the third digit.