19th Philippine Mathematical Olympiad

Qualifying Stage

22 October 2016

PART I. Choose the best answer. Each correct answer is worth two points.

1. If $27^3 + 27^3 + 27^3 = 27^x$, what is the value of x?

   (a) $\frac{10}{3}$ (b) 4 (c) 9 (d) 12

2. Let $a, b > 0$. If $|x - a| \leq a + b$, then what is the minimum value of x?

   (a) $-2a - b$ (b) $-a - b$ (c) a (d) $-b$

3. One diagonal of a rhombus is three times as long as the other. If the rhombus has an area of 54 square meters, what is its perimeter?

   (a) 9 meters (b) $12\sqrt{10}$ meters (c) 36 meters (d) $9\sqrt{5}$ meters

4. Suppose that r_1 and r_2 are the roots of the equation $4x^2 - 3x - 7 = 0$. What is the sum of the squares of the reciprocals of r_1 and r_2?

   (a) $-\frac{3}{7}$ (b) $-\frac{47}{49}$ (c) $\frac{65}{49}$ (d) $\frac{6}{7}$

5. The lengths of the sides of a triangle are 3, 5, and x. The lengths of the sides of another triangle are 4, 6, and y. If the lengths of all sides of both triangles are integers, what is the maximum value of $|x - y|$?

   (a) 2 (b) 6 (c) 7 (d) 8

6. Three circles with radii 4, 5, and 9 have the same center. If $x\%$ of the area of the largest circle lies between the other two circles, what is x to the nearest integer?

   (a) 9 (b) 11 (c) 25 (d) 33

7. Issa has an urn containing only red and blue marbles. She selects a number of marbles from the urn at random and without replacement. She needs to draw at least N marbles in order to be sure that she has at least two red marbles. In contrast, she needs three times as much in order to be sure that she has at least two blue marbles. How many marbles are there in the urn?

   (a) $4N - 4$ (b) $4N - 3$ (c) $4N - 2$ (d) $4N$

8. How many positive divisors of 30^9 are divisible by 400,000?

(a) 72 (b) 150 (c) 240 (d) 520

9. Evaluate the following sum:

$$1+\cos\frac{\pi}{3}+\cos\frac{2\pi}{3}+\cos\frac{3\pi}{3}+\cos\frac{4\pi}{3}+\ldots+\cos\frac{2016\pi}{3}$$

(a) 1 (b) 0 (c) -1 (d) $\frac{1}{2}$

10. An infinite geometric series has first term 7 and sum between 8 and 9, inclusive. Find the sum of the smallest and largest possible values of its common ratio.

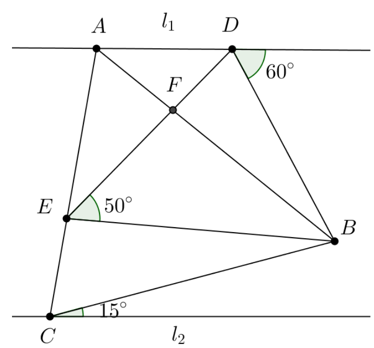
(a) 15/56 (b) 23/56 (c) 17/72 (d) 25/72

11. When $2a$ is divided by 7, the remainder is 5. When $3b$ is divided by 7, the remainder is also 5. What is the remainder when $a+b$ is divided by 7?

(a) 2 (b) 3 (c) 5 (d) 6

12. In the figure on the right (not drawn to scale), triangle ABC is equilateral, triangle DBE is isosceles with $ED = BD$, and the lines l_1 and l_2 are parallel. What is $m\angle FBE$?

(a) 30° (b) 35° (c) 40° (d) 45°



13. How many three-digit numbers have distinct digits that add up to 21?

(a) 18 (b) 24 (c) 30 (d) 36

14. A regular hexagon with area 28 is inscribed in a circle. What would the area of a square inscribed in the same circle be?

(a) $28\sqrt{3}$ (b) $\frac{56}{\sqrt{6}}$ (c) $\frac{112}{3\sqrt{3}}$ (d) $28\sqrt{6}$

15. A positive integer n is a *triangular number* if there exists some positive integer k for which it is the sum of the first k positive integers, that is, $n = 1+2+\cdots+(k-1)+k$. How many triangular numbers are there which are less than 2016?

(a) 61 (b) 62 (c) 63 (d) 64

PART II. Choose the best answer. Each correct answer is worth three points.

1. I have 2016 identical marbles. I plan to distribute them equally into one or more identical containers. How many ways can this be done if I have an unlimited number of containers?

   (a) 10 (b) 36 (c) 1008 (d) 6552

2. Suppose that the seven-digit number $159aa72$ is a multiple of 2016. What is the sum of its distinct prime divisors?

   (a) 12 (b) 17 (c) 23 (d) 36

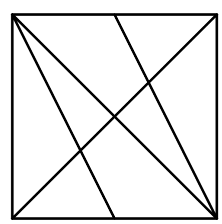
3. Let $f(x)$ be a polynomial of degree 4 with integer coefficients, leading coefficient 1, and having $\sqrt{10}+\sqrt{11}$ as one of its zeros. What is $f(1)$?

   (a) -44 (b) -40 (c) -36 (d) -21

4. If $b > 1$, find the minimum value of $\dfrac{9b^2 - 18b + 13}{b-1}$.

   (a) 0 (b) 9 (c) 12 (d) 36

5. How many triangles are there in the figure below?



   (a) 14 (b) 16 (c) 18 (d) 20

6. What is the 100th digit of the following sequence?

   1 4 9 16 25 36 49 64 81 100 ...

   (a) 6 (b) 7 (c) 8 (d) 9

7. Louie plays a game where he throws a circular coin with radius 1 unit, which falls flat entirely inside a square board having side 10 units. He wins the game if the coin touches the boundary or the interior of a circle of radius 2 units drawn at the center of the board. What is the probability that Louie wins the game?

   (a) $\dfrac{9\pi}{64}$ (b) $\dfrac{16\pi}{81}$ (c) $\dfrac{\pi}{9}$ (d) $\dfrac{9\pi}{100}$

8. Guido and David each randomly choose an integer from 1 to 100. What is the probability that neither integer is the square of the other?

   (a) 0.81 (b) 0.99 (c) 0.9919 (d) 0.9981

9. How many ordered triples of positive integers (x, y, z) are there such that $x + y + z = 20$ and two of x, y, z are odd?

   (a) 135 (b) 138 (c) 141 (d) 145

10. Suppose that $x < 0 < y < 1 < z$. Which of the following statements is true?

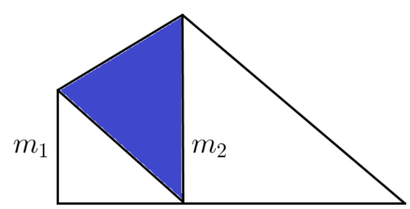
    I. $\dfrac{xz - y}{x}$ is always greater than $x + yz$

    II. $xy + z$ is always greater than $\dfrac{z - xy}{x}$

    (a) I only (b) II only (c) both I and II (d) neither I nor II

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. A paper cut-out in the shape of an isosceles right triangle is folded in such a way that one vertex meets the edge of the opposite side, and that the constructed edges m_1 and m_2 are parallel to each other (refer to figure below, which is not drawn to scale). If the length of the triangle's leg is 2 units, what is the area of the shaded region?



2. Using the numbers 1, 2, 3, 4, 5, 6, and 7, we can form $7! = 5040$ 7-digit numbers in which the 7 digits are all distinct. If these numbers are listed in increasing order, find the 2016th number in the list.

3. Let G be the set of ordered pairs (x, y) such that (x, y) is the midpoint of $(-3, 2)$ and some point on the circle $(x + 3)^2 + (y - 1)^2 = 4$. What is the largest possible distance between any two points in G?

4. Let $f(x) = \sqrt{-x^2 + 20x + 400} + \sqrt{x^2 - 20x}$. How many elements in the range of f are integers?

5. For every positive integer n, let $s(n)$ denote the number of terminal zeroes in the decimal representation of $n!$. For example, $10! = 3,628,800$ ends in two zeroes, so $s(10) = 2$. How many positive integers less than or equal to 2016 cannot be expressed in the form $n + s(n)$ for some positive integer n?

Answers to the 19th PMO Qualifying Stage

Part I. (2 points each)

1. A
2. D
3. B
4. C
5. B
6. B
7. A
8. B
9. A
10. D
11. B
12. B
13. A
14. C
15. B

Part II. (3 points each)

1. B
2. C
3. B
4. C
5. D
6. D
7. A
8. D
9. A
10. A

Part III. (6 points each)

1. $(6\sqrt{2}-8)$ sq. units
2. $3,657,421$
3. 2
4. 9
5. 401