Questionnaire No.: ______________

17^{th} Philippine Mathematical Olympiad

Qualifying Stage

11 October 2014

PART I. Choose the best answer. Each correct answer is worth two points.

1. If $f(x) = 1 + \frac{1}{x}$, find $f(f(f(1)))$.

   (a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{5}{3}$ (d) $\frac{5}{4}$

2. What is the units digit of the product of five consecutive integers?

   (a) 0 (b) 2 (c) 4 (d) 6

3. In an urn, 4/7 of the chips are red and the rest are blue. If the number of red chips is reduced by half and the number of blue chips is doubled, what is now the fraction of red chips in the urn?

   (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{2}{7}$

4. If $p = 1 + 3 + \cdots + 2011 + 2013$ and $q = 2 + 4 + \cdots + 2012 + 2014$, evaluate $q - p$.

   (a) -1008 (b) 0 (c) 1007 (d) 1008

5. Let the sum of N numbers be N. If N is added to each of these numbers, and then each is multiplied by N, what is now the sum of these N resulting numbers?

   (a) $2N^2$ (b) $N^2 + N^3$ (c) $2N^3$ (d) N^4

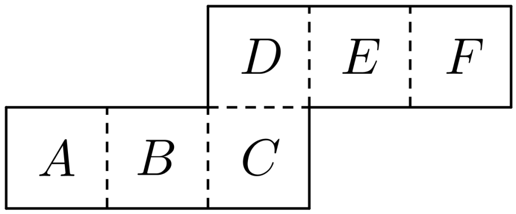
6. If $r + s - a - b = 2$ and $rs + a + b + 2 = 0$, find the value of $(r + 1)(s + 1)$.

   (a) -1 (b) 0 (c) 2 (d) 1

7. A group of 10 women and 7 men are to be arranged at random in a row. What is the probability that all of the women will be beside each other?

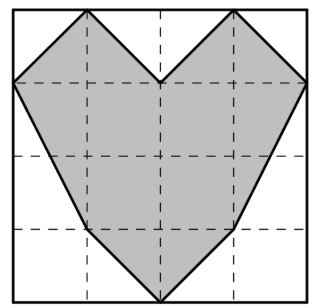
   (a) $\frac{1}{\binom{17}{10}}$ (b) $\frac{10!}{\binom{17}{10}}$ (c) $\frac{10!7!}{17!}$ (d) $\frac{10!8!}{17!}$

8. When the figure below is folded along the dotted segments to form a cube, which pair of letters will be opposite each other?



(a) (A, C) (b) (B, F) (c) (C, E) (d) (C, F)

9. In the square on the right, the dotted segments divide each side into four equal parts. If the square has area 256 square units, what is the area of the shaded octagon, in square units?



(a) 144 (b) 160 (c) 176 (d) 192

10. A huge urn contains as many marbles as needed of the following three colors: blue, red and yellow. Each blue marble weighs 42 grams, each red marble weighs 28 grams, and each yellow marble weighs 20 grams. Marbles of each color are to be distributed in three bags such that each bag contains marbles of a single color, and all bags have the same weight. What is the least number of marbles needed to do this?

(a) 36 (b) 42 (c) 46 (d) 52

11. How many zeros does $f(x) = \log(\sin x)$ have in the interval $[0, 4\pi]$?

(a) 3 (b) 2 (c) 1 (d) none

12. Suppose that x is a number such that $\dfrac{9}{a} + a \geq x$ for any positive number a. What is the largest possible value of x?

(a) 3 (b) 4 (c) 6 (d) 9

13. In the equation $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2014$, x is used infinitely many times as exponent. Which of the following is a root of the equation?

(a) $\log_{2014} 2014$ (b) $\sqrt{2014}$ (c) $\sqrt[2014]{2014}$ (d) 2014^{2014}

14. A point P is outside a circle and on the same plane as it. If the points on the circle closest and farthest from P are 4 and 16 units away, respectively, how long is a tangent segment from P to the circle?

(a) 6 (b) 8 (c) 10 (d) 12

15. How many real solutions does the equation have?

$$\log_{2014} x - \log_x 2014 = \log_{\sqrt[2014]{2014}} \sqrt[2014]{x}$$

(a) none (b) 1 (c) 2 (d) infinitely many

PART II. Choose the best answer. Each correct answer is worth three points.

1. The interior angles of a convex polygon are all $160°$, except for one which is $x°$. What is the smallest possible value of x, if the polygon has an even number of sides?

   (a) 10 (b) 20 (c) 40 (d) 80

2. For the function f, $f(2x) = x^2 + x - 2$ for all real numbers x. Let a and b be the sum and product, respectively, of the roots of the equation $f(x/2) = 4$. Find $a + b$.

   (a) 96 (b) -96 (c) 100 (d) -100

3. Find the length of the shortest path on the plane from $P(0,0)$ to $Q(2,1)$, so that any point on this path is at least one unit away from $(1,0)$, $(1,1)$, $(1,2)$ and $(2,0)$.

   (a) 2π (b) $1 + \frac{3}{2}\pi$ (c) $1 + \sqrt{2} + \pi$ (d) $\frac{11}{6}\pi$

4. Simplify:

$$\frac{1+2+3}{1+2+3+4} \times \frac{1+2+3+4+5}{1+2+3+4+5+6} \times \cdots \times \frac{1+2+\cdots+19}{1+2+\cdots+19+20}.$$

   (a) $\frac{1}{5}$ (b) $\frac{3}{20}$ (c) $\frac{1}{7}$ (d) $\frac{1}{35}$

5. An infinite geometric series has sum 2014. If the sum of their squares is also 2014, find the first term.

   (a) $\frac{2013}{2014}$ (b) $\frac{2013}{2015}$ (c) $\frac{2014}{2015^2}$ (d) $\frac{4028}{2015}$

6. Let the longest diagonal of a closed rectangular box be 6 units in length. If the lengths of its sides are all integers, find the surface area of the box in square units.

   (a) 48 (b) 64 (c) 76 (d) 96

7. For how many integers $n > 0$ will the sum of the first n positive integers be a factor of $8n^2$?

   (a) 4 (b) 5 (c) 6 (d) infinitely many

8. For which value of the constant k below will the inequality

$$9k^2(x-5)^2 - 125k^2 \geq (9+5k^2)(x^2-10x) + 225$$

have a unique solution?

(a) $\frac{1}{2014}$ (b) $\frac{3}{2}$ (c) -9 (d) 2014

9. How many values of the integer k will make the triangle with sides 6, 8 and k obtuse?

(a) 3 (b) 6 (c) 8 (d) 11

10. How many polynomials

$$x^{2014} + a_{2013}x^{2013} + a_{2012}x^{2012} + \ldots + a_2x^2 + a_1x + a_0$$

with real coefficients $a_0, a_1, \ldots, a_{2013}$ can be formed, if all its zeros are real and can only come from the set $\{1, 2, 3\}$?

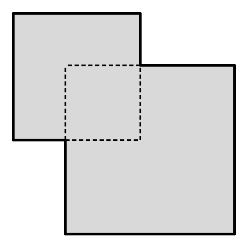
(a) 2014^3 (b) $\binom{2014}{2}$ (c) $\binom{2015}{3}$ (d) $\binom{2016}{2}$

PART III. All answers should be in simplest form. Each correct answer is worth six points.

1. Solve the equation

$$(2 \cdot 3^x)^3 + (9^x - 3)^3 = (9^x + 2 \cdot 3^x - 3)^3.$$

2. Two squares have integer side lengths which are in the ratio 4 : 3. Their intersection is also a square with integer side length. If the total area shown on the surface is equal to 5000 square units, how long is a side of the largest square?



3. How many 3-element subsets of $\{1, 2, 3, \ldots, 11, 12, 13\}$ are there for which the sum of the 3 elements is divisible by 3?

4. A sequence $a_1, a_2, a_3, \ldots$ is defined in the following manner: $a_1 = 1$, and for every integer $n \geq 2$, a_n is the nth even integer greater than a_{n-1}. Find the remainder when a_{2014} is divided by 2014.

5. Let α, β and γ be the roots of $x^3 - 4x - 8 = 0$. Find the numerical value of the expression

$$\frac{\alpha+2}{\alpha-2} + \frac{\beta+2}{\beta-2} + \frac{\gamma+2}{\gamma-2}.$$