19^{th} Philippine Mathematical Olympiad

National Stage, Written Phase

21 January 2017

Time: 4.5 hours *Each item is worth 7 points.*

1. Given $n \in \mathbb{N}$, let $\sigma(n)$ denote the sum of the divisors of n and $\varphi(n)$ denote the number of positive integers $m \leq n$ for which $\gcd(m, n) = 1$. Show that for all $n \in \mathbb{N}$,

$$\frac{1}{\varphi(n)} + \frac{1}{\sigma(n)} \geq \frac{2}{n}$$

and determine when equality holds.

Solution. We note that equality holds for $n = 1$. We prove the inequality when $n > 1$ and show that it is strict in this case.

By the AM-GM inequality, $\frac{1}{\varphi(n)} + \frac{1}{\sigma(n)} \geq \frac{2}{\sqrt{\varphi(n)\sigma(n)}}$. Hence, we need only show that $\varphi(n)\sigma(n) < n^2$, or equivalently, $\frac{\varphi(n)\sigma(n)}{n^2} < 1$. We note that $f(n) := \frac{\varphi(n)\sigma(n)}{n^2}$ is multiplicative, i.e., $f(mn) = f(m)f(n)$ whenever $\gcd(m, n) = 1$, and so it suffices to show that $f(n) < 1$ when $n = p^k$ for some prime p and some integer $k \geq 1$. However,

$$\begin{aligned} f(p^k) &= \frac{\varphi(p^k)\sigma(p^k)}{(p^k)^2} = \frac{\varphi(p^k)}{p^k} \cdot \frac{\sigma(p^k)}{p^k} = \frac{(p-1)p^{k-1}}{p^k} \cdot \frac{1}{p^k} \sum_{i=0}^{k} p^i \\ &= \frac{p-1}{p} \cdot \sum_{i=0}^{k} \frac{1}{p^{k-i}} = \frac{p-1}{p} \cdot \sum_{i=0}^{k} \frac{1}{p^i} = \frac{p-1}{p} \cdot \frac{1 - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}} \\ &< \frac{p-1}{p} \cdot \frac{1}{1 - \frac{1}{p}} = 1. \end{aligned}$$

This completes the proof.

2. Find all positive real numbers $a, b, c \leq 1$ such that

$$\min\left\{\sqrt{\frac{ab+1}{abc}}, \sqrt{\frac{bc+1}{abc}}, \sqrt{\frac{ac+1}{abc}}\right\} = \sqrt{\frac{1-a}{a}} + \sqrt{\frac{1-b}{b}} + \sqrt{\frac{1-c}{c}}$$

Modified from Problem 4 of the 2013 USA Math Olympiad

Solution. Let $r, s, t \geq 0$ such that

$$a = \frac{1}{1+r^2}, \ b = \frac{1}{1+s^2}, \ c = \frac{1}{1+t^2}.$$

Also, WLOG, suppose $t = \min\{r, s, t\}$. The required equation can then be rewritten as

$$\sqrt{(1+t^2)\{1+(1+r^2)(1+s^2)\}} = r+s+t.$$

By Cauchy-Schwarz, $(r+s+t)^2 \leq ([r+s]^2+1)(1+t^2)$, and so we have

$$(1+r^2)(1+s^2) \leq (r+s)^2,$$

which is equivalent to $(rs-1)^2 \leq 0$. Only equality is true here; hence $rs = 1$ and all preceding inequalities become equations, and so $t(r+s) = 1$. Now, conversely, if $rs = 1$ and $t(r+s) = 1$, with $t = \dfrac{1}{r+s}$ still less than both $\dfrac{1}{r} = s$ and $\dfrac{1}{s} = r$, the condition of the problem is still satisfied. Therefore, the solutions are

$$a = \frac{1}{1+r^2}, \ b = \frac{1}{1+\frac{1}{r^2}}, \ c = \frac{(r+\frac{1}{r})^2}{1+(r+\frac{1}{r})^2}$$

and permutations of these.

3. Each of the numbers in the set $A = \{1, 2, ..., 2017\}$ is colored either red or white. Prove that for $n \geq 18$, there exists a coloring of the numbers in A such that any of its n-term arithmetic sequences contains both colors.

Inspired by Problem 891 from *Putnam and Beyond*, T. Andreescu and R. Gelca, Springer (2007)

Solution. It suffices to show that for $n \geq 18$, the total number of colorings (without restriction) exceeds those that make some n-term arithmetic sequence monochromatic.

There are 2^{2017} colorings of a set with 2017 elements. The number of colorings that make a fixed n-term sequence monochromatic is $2 \cdot 2^{2017-n} = 2^{2018-n}$, since the terms not in the sequence can be colored without restriction, while those in the sequence can be colored either all red or all white.

We now find the number of n-term arithmetic sequences that can be obtained from A. Such a sequence $a, a+r, \ldots, a+(n-1)r$ is completely determined by the first term a and common ratio r, subject to the constraint $a+(n-1)r \leq 2017$. For each value of a, there are $\left\lfloor \frac{2017-a}{n-1} \right\rfloor$

sequences that start with a. This means that the number of arithmetic sequences does not exceed

$$\sum_{a=1}^{2017} \frac{2017-a}{n-1} = \frac{2016 \cdot 2017}{2(n-1)}.$$

Therefore, the total number of colorings that make at least one arithmetic sequence monochromatic does not exceed

$$2^{2018-n} \cdot \frac{2016 \cdot 2017}{2(n-1)}.$$

But for $n \geq 18$,

$$\begin{aligned} 2^{2018-n} \cdot \frac{2016 \cdot 2017}{2(n-1)} &\leq 2^{2018-n} \cdot \frac{2048 \cdot 2048}{2(n-1)} \\ &= \frac{2^{2039-n}}{n-1} \leq \frac{2^{2021}}{17} < 2^{2017}. \end{aligned}$$

4. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centers at C_1 and C_2, respectively, intersect at two distinct points A and B. Points P and Q are varying points on $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, such that P, Q and B are collinear and B is always between P and Q. Let lines PC_1 and QC_2 intersect at R, let I be the incenter of $\triangle PQR$, and let S be the circumcenter of $\triangle PIQ$. Show that as P and Q vary, S traces an arc of a circle whose center is concyclic with A, C_1 and C_2.

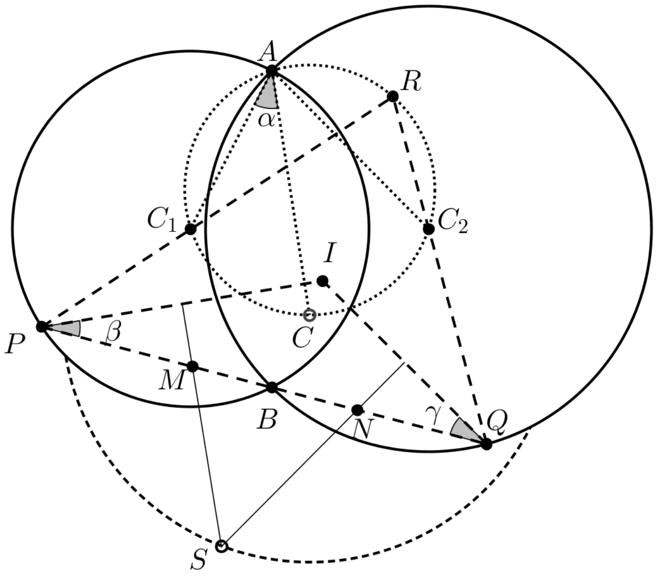
<u>*Solution.*</u> Let C be the intersection of the circle through C_1, C_2 and A, and the bisector of $\angle C_1AC_2$. It suffices to show that $CS = CA$. Let $\angle C_1AC_2 = C_1BC_2 = 2\alpha$ (fixed), $\angle C_1PB = \angle C_1BP = 2\beta$ and $\angle C_2QB = \angle C_2BQ = 2\gamma$. Since

$$\begin{aligned} \angle C_1RC_2 &= 180^\circ - \angle C_1PB - \angle C_2QB = 180^\circ - \angle C_1BP - \angle C_2BQ \\ &= \angle C_1BC_2 = 2\alpha, \end{aligned}$$

then R is concyclic with A, C_1, C_2 and C. Also, since RI bisects $\angle C_1RC_2$, then $\angle C_1RI = \alpha = \angle C_1RC$, so C is collinear with R and I. We note too that by considering the angles of $\triangle PQR$, $\alpha + \beta + \gamma = 90^\circ$.

Let the perpendicular bisectors of PI and QI meet PQ at M and N, respectively. Since $SP = SI = SQ$, then $\angle SIM = \angle SPM = \angle SQN = \angle SIN$ and so SI bisects $\angle MIN$, where

$$\begin{aligned} \angle MIN &= 180^\circ - (\angle MIP + \angle MPI) - (\angle NIQ + \angle NQI) \\ &= 180^\circ - 2\beta - 2\gamma = 2\alpha. \end{aligned}$$



Thus, $\alpha = \angle SIM = \angle SIN$. Consequently, S is collinear with R, I and C since $\angle PIS = \angle PIM + \angle SIM = \beta + \alpha = \angle IPR + \angle IRP = 180^\circ - \angle RIP$. Furthermore, since $\angle SQP = \angle SIN = \alpha = \angle SRP$, then $PRQS$ is cyclic.

By Ptolemy's Theorem applied to C_1AC_2C,

$$\begin{aligned} CA \cdot C_1C_2 &= CC_1 \cdot AC_2 + CC_2 \cdot AC_1 = CC_1(AC_1 + AC_2) \\ &= CC_1(PC_1 + QC_2) = CC_1(RP - RC_1 + RQ - RC_2) \\ &= CC_1(RP + RQ) - CC_1(RC_1 + RC_2). \end{aligned}$$

We use Ptolemy's Theorem next on C_1RC_2C and $PRQS$. Since $\triangle C_1CC_2 \sim \triangle PSQ$ (both are isosceles with base angle α),

$$\begin{aligned} CA &= \frac{SP}{PQ}(RP + RQ) - \frac{CC_1}{C_1C_2}(RC_1 + RC_2) \\ &= \frac{SQ \cdot RP + SP \cdot RQ}{PQ} - \frac{CC_2 \cdot RC_1 + CC_1 \cdot RC_2}{C_1C_2} \\ &= \frac{RS \cdot PQ}{PQ} - \frac{RC \cdot C_1C_2}{C_1C_2} = RS - RC = CS. \end{aligned}$$