19th Philippine Mathematical Olympiad

Area Stage

19 November 2016

PART I. (3 points each)

1. 504
2. 6048
3. 7.2 units
4. 597/2 or 298.5
5. $2\sqrt{3}$
6. $\dfrac{\pi}{4}$
7. $\dfrac{1}{48}$
8. $\dfrac{15}{2}$
9. 9375
10. $y = -7x + \dfrac{28}{5}$
11. 6
12. 391
13. $\dfrac{48 - 32\sqrt{2}}{9}$
14. 5^{1008}
15. 470
16. $2015 - k$
17. 3
18. 128
19. $\dfrac{336}{325}$
20. $1 + \sqrt{2}$

PART II. (10 points each)

1. The given system can be expressed as follows:

$$\begin{cases} \dfrac{x}{x^2y^2-1} - \frac{1}{x} &= 4 \\ \dfrac{x^2y}{x^2y^2-1} + y &= 2 \end{cases} \Rightarrow \begin{cases} \dfrac{x^2}{x^2y^2-1} - 1 &= 4x \\ \dfrac{x^2}{x^2y^2-1} + 1 &= \dfrac{2}{y} \end{cases}$$

We then have

$$4x + \frac{2}{y} = \frac{2x^2}{x^2y^2-1} \Rightarrow 2x + \frac{1}{y} = \frac{x^2}{x^2y^2-1}$$

and

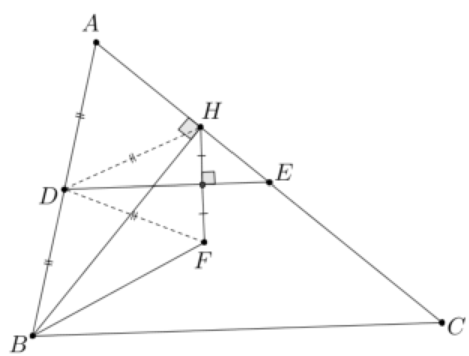
$$4x - \frac{2}{y} = -2 \Rightarrow 2x - \frac{1}{y} = -1,$$

which gives us

$$
\begin{aligned}
\left(2x+\frac{1}{y}\right)\left(2x-\frac{1}{y}\right) &= \frac{-x^2}{x^2y^2-1}\\
4x^2-\frac{1}{y^2} &= \frac{-x^2}{x^2y^2-1}\\
\frac{4x^2y^2-1}{y^2} &= \frac{-x^2}{x^2y^2-1}\\
\left(4x^2y^2-1\right)\left(x^2y^2-1\right) &= -x^2y^2\\
4x^4y^4-5x^2y^2+1 &= -x^2y^2\\
4x^4y^4-4x^2y^2+1 &= 0\\
\left(2x^2y^2-1\right)^2 &= 0 \Rightarrow x^2y^2=\frac{1}{2} \Rightarrow \boxed{xy=\pm\frac{1}{\sqrt{2}}}
\end{aligned}
$$

2. This problem is taken from the ***2015 Iranian Geometry Olympiad***.

   Solution 1. Let O be the circumcenter of ΔABC. Since $\angle OBA = 90^\circ - \angle C$, it suffices to show that $\angle FBA = 90^\circ - \angle C$.



   Note that $AD = BD = DH$ and $DH = DF$. Therefore, quadrilateral $AHFB$ is cyclic (with circumcenter D), and so $\angle FBA = \angle FHE = 90^\circ - \angle DEH$. Since DE is parallel to BC, $\angle DEH = \angle C$, and $\angle FBA = 90^\circ - \angle C$. □

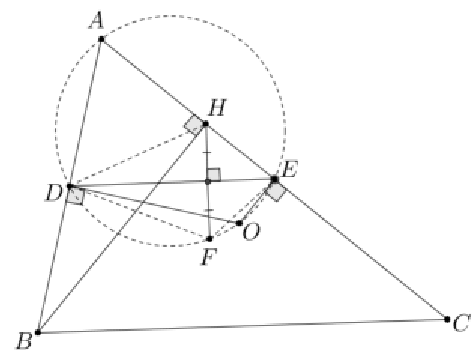
   Solution 2. As before, denote by O the circumcenter of ΔABC. Then the quadrilateral $ADOE$ is cyclic. Also, we know that $AD = HD = DB$, therefore,

$$\angle A = \angle DHA = 180^\circ - \angle DHE = 180^\circ - \angle DFE$$

   Therefore, $ADFE$ is cyclic. Since $ADFOE$ is cyclic, $DFOE$ is also cyclic, and

$$\angle C = \angle DEA = \angle DEF = \angle DOF$$

   On the other hand, $\angle C = \angle DOB$, so $\angle DOF = \angle DOB$, therefore B, F, and O are collinear. □



3. Consider $h(x) := g(x) + x + 1$. We have that, for m, n coprime and greater than 1,

$$\begin{aligned} h(m)h(n) &= (g(m) + m + 1)(g(n) + n + 1) \\ &= g(m)g(n) + (n+1)g(m) + (m+1)g(n) + mn + m + n + 1 \\ &= g(mn) + mn + 1 \\ &= h(mn). \end{aligned}$$

Repeating this, we find that more generally, if $m_1, m_2, \ldots, m_k$ are pairwise coprime positive integers all greater than 1,

$$h\left(\prod_{i=1}^{k} m_i\right) = \prod_{i=1}^{k} m_i.$$

Hence, it suffices to consider h, and thus g, only on prime powers. Since

$$g(p^n) > g(p^{n-1}) > \cdots > g(p) > g(1) \geq 1,$$

we have $g(p^n) \geq n + 1$. Indeed, taking $g(1) = 1$, $g(p^n) = n + 1$ gives us a well-defined function g on $\mathbb{N}$. To solve for $g(2016)$, we solve for $h(2016)$ first, noting that $2016 = 2^5 \cdot 3^2 \cdot 7^1$:

$$\begin{aligned} h(2016) &= h(2^5)h(3^2)h(7^1) \\ &= (7 + 2^5)(4 + 3^2)(3 + 7^1) \\ &= 5070 \end{aligned}$$

and so $g(2016) = 5070 - 2017 = \boxed{3053}$. This is the minimum possible value of $g(2016)$. $\square$