18^{th} Philippine Mathematical Olympiad

National Stage, Written Phase

23 January 2016

Time: 4.5 hours *Each item is worth 8 points.*

1. The operations below can be applied on any expression of the form $ax^2 + bx + c$.

   (I) If $c \neq 0$, replace a by $4a - \dfrac{3}{c}$ and c by $\dfrac{c}{4}$.

   (II) If $a \neq 0$, replace a by $-\dfrac{a}{2}$ and c by $-2c + \dfrac{3}{a}$.

   (III_t) Replace x by $x - t$, where t is an integer. (Different values of t can be used.)

   Is it possible to transform $x^2 - x - 6$ into each of the following by applying some sequence of the above operations?

   (a) $5x^2 + 5x - 1$ (b) $x^2 + 6x + 2$

   Solution: Each operation changes the discriminant D of $ax^2 + bx + c$ into

   (I) $D' = b^2 - 4\left(4a - \frac{3}{c}\right)\left(\frac{c}{4}\right) = b^2 - 4ac + 3 = D + 3$

   (II) $D' = b^2 - 4\left(-\frac{a}{2}\right)\left(-2c + \frac{3}{a}\right) = b^2 - 4ac + 6 = D + 6$

   (III_t) $D' = (b - 2at)^2 - 4a(at^2 - bt + c) = b^2 - 4ac = D$ Note: $b \to b - 2at$

   We point out that it suffices to refer to a single application of (III) instead of successive applications since for instance, applying (III_{t_1}) and then (III_{t_2}) is equivalent to applying ($\text{III}_{t_1+t_2}$). Furthermore, applying (III) retains the quadratic coefficient. It also doesn't change the parity of the linear coefficient if a and b are integers since (III_t) changes this coefficient from b to $b - 2at$.

   Suppose we apply on $p(x) = x^2 - x - 6$, in some order, (I) m times and (II) n times, possibly along with (III). The discriminant of p is 25 while that of $q(x) = 5x^2 + 5x - 1$ is 45. Therefore, $25 + 3m + 6n = 45$, so $3m + 6n = 20$. This has no integer solutions for (m, n) since 20 is not divisible by 3. Thus, it is not possible to transform p into q.

   Next, suppose it is possible to transform p into r. Since $r(x) = x^2 + 6x + 2$ has discriminant 28, we have $25 + 3m + 6n = 28$. Then $m + 2n = 1$, so $(m, n) = (1, 0)$. This means (I) is applied once and (II) never.

- Suppose (I) is applied first on p, changing its quadratic coefficient to $\frac{9}{2}$. Applying (III) retains this, but the quadratic coefficient of r is 1.
- Suppose (III) is applied first, changing the coefficients $(1, -1, -6)$ of p into $(1, n, c_0)$, n odd, $c_0 \in \mathbb{Z}$. Applying (I) next transforms the first two coefficients to $\left(4 - \frac{3}{c_0}, n\right)$. Applying (III) will not change the quadratic coefficient, so we may suppose $4 - \frac{3}{c_0} = 1$, i.e. $c_0 = 1$. Since n is odd, while the linear coefficient of r is 6, we cannot stop yet. However, the only operation we can apply now is (III), which will not change the parity (odd) of the linear coefficient.

Thus, it is not possible to change p into r.

2. Prove that the arithmetic sequence $5, 11, 17, 23, 29, \ldots$ contains infinitely many primes.

Solution: The terms of the sequence are all of the form $6n + 5$. By contradiction, suppose there is a largest prime p in the sequence. Let $q = (2 \cdot 3 \cdot 5 \cdots p) - 1$, one less than the product of all primes not exceeding p. This is a number of the form $6n + 5$, and so should be in the sequence.

If q is a prime, since $q > p$, we have a contradiction (p is the largest prime in the sequence).

Suppose q is composite. None of the primes up to p is a divisor of q. Thus, all the prime factors of q are greater than p. These primes, being greater than p, are not in the sequence, and so should each be of the form $6n + 1$. But then their product should also be of the form $6n + 1$. Contradiction, since q is of the form $6n + 5$.

3. Let n be any positive integer. Prove that

$$\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{3/4}} > 2 - \frac{2}{\sqrt{n+1}}.$$

Solution: By the AM-GM Inequality, we have for each i,

$$\frac{(i+1)\sqrt{i} + i\sqrt{i+1}}{2} > \sqrt{(i+1)^{3/2} \cdot i^{3/2}} = (i^2 + i)^{3/4}.$$

Note that this inequality is strict, as $(i+1)\sqrt{i}$ cannot be equal to $i\sqrt{i+1}$. Thus,

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{(i^2 + i)^{3/4}} &> \sum_{i=1}^{n} \frac{2}{(i+1)\sqrt{i} + i\sqrt{i+1}} = 2\sum_{i=1}^{n} \frac{1}{\sqrt{i(i+1)}\left(\sqrt{i+1} + \sqrt{i}\right)} \\
&= 2\sum_{i=1}^{n} \frac{\sqrt{i+1} - \sqrt{i}}{\sqrt{i(i+1)}} = 2\sum_{i=1}^{n} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}}\right) = 2\left(1 - \frac{1}{\sqrt{n+1}}\right).
\end{aligned}$$

4. Two players, A (first player) and B, take alternate turns in playing a game using 2016 chips as follows: *the player whose turn it is, must remove s chips from the remaining pile of chips, where $s \in \{2, 4, 5\}$.* No one can skip a turn. The player who at some point is unable to make a move (cannot remove chips from the pile) loses the game. Who among the two players can force a win on this game?

   Solution: We call the remaining number of chips a winning position if there exists at least one move such that the player (whose turn it is) can force a win. The remaining number of chips is a losing position if any move by the player will give the opponent a chance to force a win, or a winning position. Thus, the number of chips is a winning position if there is a move that will give the opponent a losing position.

   - Clearly, 0 and 1 are losing positions: you cannot make any legal move here.
   - However, 2, 4 and 5 are winning positions: take all the remaining chips and win.
   - 3 is a winning position: the only move you can make is to remove 2 chips, leaving your opponent with 1 chip, so he loses.
   - $6 = 5 + 1$ is also a winning position: take 5 chips, causing your opponent to lose.
   - 7 and 8 are losing positions: any move will give the opponent a winning position: $7 = 2 + 5 = 4 + 3 = 5 + 2$ and $8 = 2 + 6 = 4 + 4 = 5 + 3$.
   - Positions 9 to 13 are winning positions since you can bring the game to a losing position: $9 = 2 + 7$, $10 = 2 + 8$, $11 = 4 + 7$, $12 = 5 + 7$, $13 = 5 + 8$.

   We conjecture that for nonnegative integers k, $7k$ and $7k + 1$ are losing positions and $7k + 2$, $7k + 3$, $7k + 4$, $7k + 5$ and $7k + 6$ are winning positions. We have shown this to be true when $k = 0$ and when $k = 1$. Now we proceed by mathematical induction. We assume that $7k$ and $7k + 1$ are losing positions and $7k + 2$, $7k + 3$, $7k + 4$, $7k + 5$ and $7k + 6$ are winning positions, where $k \geq 0$. We want to show that $7k + 7$ and $7k + 8$ are losing positions while $7k + 9$, $7k + 10$, $7k + 11$, $7k + 12$, $7k + 13$ are winning positions.

   - $7k+7$ and $7k+8$ are losing positions since any move will yield a winning position: $7k + 7 = 2 + (7k + 5) = 4 + (7k + 3) = 5 + (7k + 2)$ and $7k + 8 = 2 + (7k + 6) = 4 + (7k + 4) = 5 + (7k + 3)$.
   - $7k + 9$ is a winning position: $7k + 9 = 2 + (7k + 7)$.
   - $7k + 10$ is a winning position: $7k + 10 = 2 + (7k + 8)$.
   - $7k + 11$ is a winning position: $7k + 11 = 4 + (7k + 7)$.
   - $7k + 12$ is a winning position: $7k + 12 = 4 + (7k + 8)$.
   - $7k + 13$ is a winning position: $7k + 13 = 5 + (7k + 8)$.

   Since $2016 = 7(288) + 0$, 2016 is a losing position. Thus, the second player can force a win under this game.

5. Pentagon $ABCDE$ is inscribed in a circle. Its diagonals AC and BD intersect at F. The bisectors of $\angle BAC$ and $\angle CDB$ intersect at G. Let AG intersect BD at H, let DG intersect AC at I, and let EG intersect AD at J. If $FHGI$ is cyclic and

$$JA \cdot FC \cdot GH = JD \cdot FB \cdot GI,$$

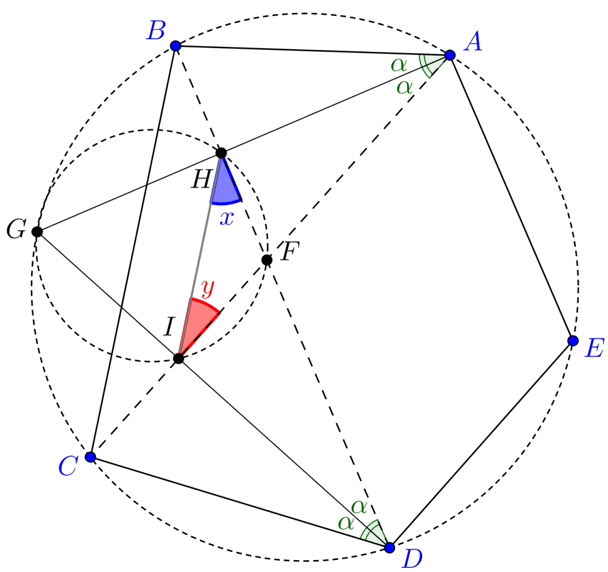
prove that G, F and E are collinear.

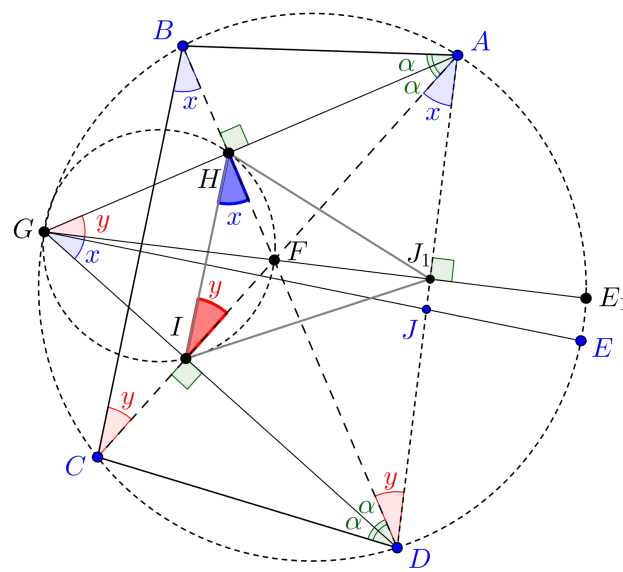
Solution: Since $\angle BAC$ and $\angle BDC$ subtend the same arc, we can let $\alpha = \angle BAG = \angle GAC = \angle CDG = \angle GDB$. Since $\angle BAG = \angle BDG$, then G is a point on the circumcircle.

Let $x = \angle FHI$ and $y = \angle FIH$. Since $AHID$ is cyclic ($\angle HAI = \angle IDH = \alpha$), then $\angle IAD = x$ and $\angle HDA = y$. Since $ABCD$ is cyclic, we also have $\angle FBC = x$ and $\angle FCB = y$.

Since $FHGI$ is cyclic, then $\angle FGI = x$ and $\angle FGH = y$. By adding the angles of $\triangle AGD$, we get as a result: $x + y + \alpha = 90°$.

Extend GF, intersecting AD at J_1, and the circumcircle of the pentagon at E_1. One consequence we get is that $GJ_1 \perp AD$ (because the highlighted angles of $\triangle AJ_1G$, $\alpha + x + y$, already add up to $90°$). Similarly, $DH \perp AG$ and $AI \perp DG$.





The equation now implies

$$\frac{JA}{JD} = \frac{FB}{FC} \cdot \frac{GI}{GH} = \frac{FH}{FI} \cdot \frac{GI}{GH} = \frac{FH/GH}{FI/GI} = \frac{FJ_1/J_1D}{FJ_1/J_A} = \frac{J_1A}{J_1D}.$$

This forces $J = J_1$ and so $E = E_1$. Therefore, G, F and E are collinear.