18th Philippine Mathematical Olympiad

Area Stage

14 November 2015

PART I. Give the answer in the simplest form that is reasonable. No solution is needed. Figures are not drawn to scale. Each correct answer is worth three points.

1. Marc and Jon together have 66 marbles although Marc has twice as many marbles as Jon. Incidentally, Jon found a bag of marbles which enabled him to have three times as many marbles as Mark. How many marbles were in the bag that Jon found?

2. A camera's aperture determines the size of the circular opening in the lens that allows light in. If we want to allow twice as much light in, what should be the ratio of the new radius to the current radius?

3. Determine all values of $k \in \mathbb{R}$ for which the equation

$$\frac{4(2015^x) - 2015^{-x}}{2015^x - 3(2015^{-x})} = k$$

admits a real solution.

4. The points $(3, m)$, (x_1, y_1) and (x_2, y_2) are on the graph of the function $f(x) = \log_a x$. If $y_1 + y_2 = 2m$, find the value of $x_1 x_2$.

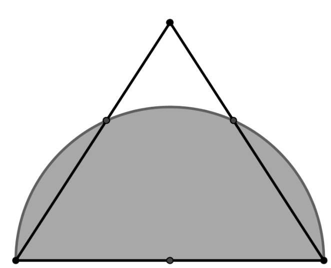
5. Let a, b, and c be three consecutive even numbers such that $a > b > c$. What is the value of $a^2 + b^2 + c^2 - ab - bc - ac$?

6. Evaluate

$$\prod_{\theta=1}^{89} (\tan\theta^\circ \cos 1^\circ + \sin 1^\circ).$$

7. Find the sum of all the prime factors of 27,000,001.

8. Refer to the figure on the right. A side of an equilateral triangle is the diameter of the given semi-circle. If the radius of the semi-circle is 1, find the area of the unshaded region inside the triangle.



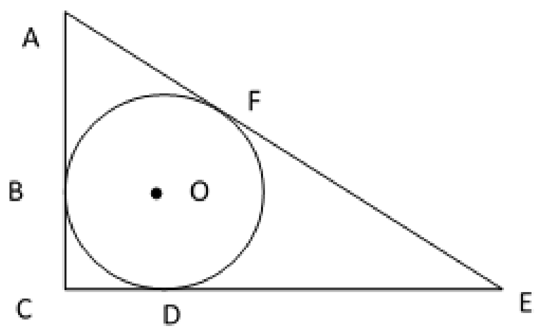
(Figure is not drawn to scale.)

9. How many ways can you place 10 identical balls in 3 baskets of different colors if it is possible for a basket to be empty?

10. Find the largest number N so that

$$\sum_{n=5}^{N} \frac{1}{n(n-2)} < \frac{1}{4}.$$

11. Refer to the figure below. If circle O is inscribed in the right triangle ACE as shown below, and if the length of AB is twice the length of BC, find the length of CE if the perimeter of the right triangle is 36 units.



12. Find all real solutions to the system of equations

$$\begin{cases} x(y-1)+y(x+1) &= 6, \\ (x-1)(y+1) &= 1. \end{cases}$$

13. Find all real numbers a and b so that for all real numbers x,

$$2\cos^2\left(x+\frac{b}{2}\right) - 2\sin\left(ax-\frac{\pi}{2}\right)\cos\left(ax-\frac{\pi}{2}\right) = 1.$$

14. Let P be the product of all prime numbers less than 90. Find the largest integer N so that for each $n \in \{2, 3, 4, ..., N\}$, the number $P+n$ has a prime factor less than 90.

15. In how many ways can the letters of the word ALGEBRA be arranged if the order of the vowels must remain unchanged?

16. The lengths of the sides of a rectangle are all integers. Four times its perimeter is numerically equal to one less than its area. Find the largest possible perimeter of such a rectangle.

17. Find the area of the region bounded by the graph of $|x|+|y| = \frac{1}{4}|x+15|$.

18. Given $f(1-x)+(1-x)f(x) = 5$ for all real number x, find the maximum value that is attained by $f(x)$.

19. The amount 4.5 is split into two nonnegative real numbers uniformly at random. Then each number is rounded to its nearest integer. For instance, if 4.5 is split into $\sqrt{2}$ and $4.5 - \sqrt{2}$, then the resulting integers are 1 and 3, respectively. What is the probability that the two integers sum up to 5?

20. Let s_n be the sum of the digits of a natural number n. Find the smallest value of $\frac{n}{s_n}$ if n is a four-digit number.

PART II. Show your solution to each problem. Each complete and correct solution is worth ten points.

1. The 6-digit number $739ABC$ is divisible by 7, 8, and 9. What values can A, B, and C take?

2. The numbers from 1 to 36 can be written in a counterclockwise spiral as follows:

[31]	30	29	28	27	26
32	**[13]**	12	11	10	25
33	14	**[3]**	2	9	24
34	15	4	**[1]**	8	23
35	16	5	6	**[7]**	22
36	17	18	19	20	**[21]**

In the figure above, all the terms on the diagonal beginning from the upper left corner have been enclosed in a box, and these entries sum up to 76.

Suppose this spiral is continued all the way until 2015, leaving an incomplete square. Find the sum of all the terms on the diagonal beginning from the upper left corner of the resulting (incomplete) square.

3. Point P on side BC of triangle ABC satisfies

$$|BP| : |PC| = 2 : 1.$$

Prove that the line AP bisects the median of triangle ABC drawn from vertex C.