18th Philippine Mathematical Olympiad

Area Stage

14 November 2015

PART I. (3 points each)

1. 110 marbles
2. $\sqrt{2}$
3. $k \in (-\infty, \frac{1}{3}) \cup (4, +\infty)$
4. 9
5. 12
6. $\csc 1^\circ$ or $\sec 89^\circ$ or equivalent
7. 652
8. $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$
9. 66
10. 24
11. 12
12. $(4/3, 2), (-2, -4/3)$
13. $a = 1$ and $b = -\frac{3\pi}{2} + 2k\pi$, or $a = -1$ and $b = \frac{3\pi}{2} + 2k\pi$ where $k \in \mathbb{Z}$
14. 96
15. 840
16. 164 units
17. 30
18. 5
19. 4/9
20. $\dfrac{1099}{19}$

PART II. (10 points each)

1. Since $\gcd(7, 8, 9) = 1$, then $739ABC$ is divisible by $7, 8$, and 9 iff it is divisible by $7 \cdot 8 \cdot 9 = 504$. Note that the only integers between 739000 and 739999 which are divisible by 504 are 739368 and 739872. So, $(A, B, C) \in \{(3, 6, 8), (8, 7, 2)\}$.

2. Solution 1:

   The closest perfect square to 2015 is $2025 = 45^2$ which means that only the rightmost side will be incomplete while the required diagonal would still have a total of 45 entries.

   Looking at the values on the diagonal, we see that the numbers on the diagonal above 1 have a common second dierence. This suggests that this sequence satisfies a quadratic function of the form $f(n) = an^2 + bn + c$. Since $f(1) = 1$, $f(2) = 3$, $f(3) = 13$, solving a simple system of three equations gives us $f(n) = 4n^2 - 10n + 7$, $1 \leq n \leq 23$. On the other hand, the numbers on the diagonal below 1 also have a common second difference. This gives a sequence $g(n) = dn^2 + en + f$ with $g(1) = 1$, $g(2) = 7$, and $g(3) = 21$. By solving a similar system as above, we obtain $g(n) = 4n^2 - 6n + 3$, where $1 \leq n \leq 23$. Taking the sum of these two sequences of numbers, we have

$$\begin{aligned}
\sum_{n=1}^{23} [f(n) + g(n)] &= \sum_{n=1}^{23} \left(8n^2 - 16n + 10\right) \\
&= 8\sum_{n=1}^{23} n^2 - 16\sum_{n=1}^{23} n + \sum_{n=1}^{23} 10 \\
&= 8\left[\frac{(23)(24)(47)}{6}\right] - 168\left[\frac{(23)(24)}{2}\right] + 10(23) \\
&= 30,406
\end{aligned}$$

Since 1 is counted twice, the required sum must be $30,406 - 1 = 30,405$.

Solution 2:

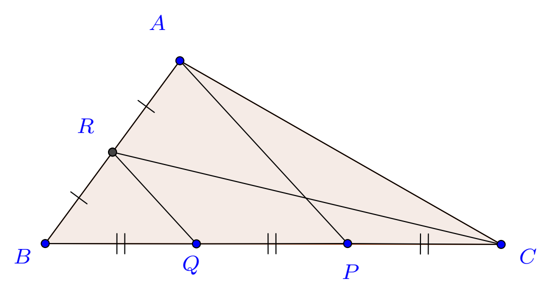
Filling the square with a few more numbers enables us to see that the boxed numbers

$$1, 3, 7, 13, 21, 31, \ldots, 1981$$

satisfy the recurrence relation $a_1 = 1$ and $(\forall n \in \mathbb{N})\ a_{n+1} = a_n + 2n$. The associated homogeneous recurrence relation is solved by $a_n^{(h)} \equiv 1$. Testing a particular solution of the form $a_n^{(p)} = n(cn + d)$, we see that $c = 1$ and $d = -1$. Therefore, the solution to the nonhomogeneous recurrence relation is $a_n = n^2 - n + 1$. The last boxed number 1981 corresponds to $n = 45$. Therefore,

$$\sum_{n=1}^{45}(n^2 - n + 1) = \frac{45 \cdot 46 \cdot 91}{6} - \frac{45 \cdot 46}{2} + 45 = 30,405.$$

3. Solution 1: Let Q be the midpoint of line segment BP. The conditions of the problem imply $|BQ| = |QP| = |PC| = \frac{1}{3}|BC|$. Let R be the midpoint of line segment AB. Then RQ is a midline of ABP. Consequently, $RQ \,||\, AP$. Ray AP bisects side CQ of triangle CRQ while being parallel to side RQ of this triangle. Thus AP extends the midline of triangle CRQ and bisects therefore also its side CR. But line segment CR is the median of triangle ABC drawn from vertex C.



Solution 2: Let R be the midpoint of segment AB. Choose point D on ray AC beyond point C such that $|AC| = |CD|$. Then BC is a median of triangle ABD. As $|BP| : |PC| = 2 : 1$, point P is the intersection point of medians of triangle ABD. Thus AP lies entirely on the other median of triangle ABD, i.e., ray AP bisects the segment BD. As CR is the midline of triangle ABD, we have $CR \,||\, BD$, implying that ray AP also bisects the segment CR. But this is the median of triangle ABC drawn from vertex C.