16th Philippine Mathematical Olympiad

Qualifying Stage

12 October 2013

PART I. Each correct answer is worth two points.

1. Simplify $\frac{5}{6} - \frac{7}{12} + \frac{9}{20} - \frac{11}{30} + \frac{13}{42} - \frac{15}{56}$.

   (a) $\frac{3}{8}$ (b) $\frac{7}{12}$ (c) $\frac{5}{14}$ (d) $\frac{9}{14}$

2. Brad and Angelina each tipped their waiter 50 pesos. Brad tipped 4% of his bill while Angelina tipped 10% of her bill. What is the total bill of the two?

   (a) 1500 pesos (b) 1750 pesos (c) 1250 pesos (d) 2250 pesos

3. What is the probability of getting a sum of 10 when rolling three fair six-sided dice?

   (a) $\frac{1}{6}$ (b) $\frac{1}{8}$ (c) $\frac{1}{9}$ (d) $\frac{1}{12}$

4. Eighteen 1 cm $\times$ 1 cm square tiles are arranged to form a rectangle, with no overlaps and without leaving gaps in the interior. Which of the following is NOT a possible perimeter of the rectangle?

   (a) 18 cm (b) 22 cm (c) 24 cm (d) 38 cm

5. Consider the sum $S = x! + \sum_{i=0}^{2013} i!$, where x is a one-digit nonnegative integer. How many possible values of x are there so that S is divisible by 4?

   (a) 2 (b) 3 (c) 4 (d) 5

6. Find the number of integer solutions less than 5 that satisfy the inequality $(x^3 + 4x^2)(x^2 - 3x - 2) \leq (x^3 + 4x^2)(2x^2 - 6)$.

   (a) 2 (b) 3 (c) 4 (d) 5

7. For which m does the equation $\dfrac{x-1}{x-2} = \dfrac{x-m}{x-6}$ have no solution in x?

   (a) 6 (b) 2 (c) 8 (d) 5

8. What is the remainder when 2^{201} is divided by 7?

   (a) 1 (b) 4 (c) 2 (d) 3

9. Let $b_0 = 2$ and $b_1 = 1$, and $b_{n+1} = b_{n-1} + b_n$. Which of the following digits is the last to appear in the units position of the sequence $\{b_n\}$?

   (a) 6 (b) 7 (c) 9 (d) 0

10. $3x - b = 1$ and $bx - 5 = -4$ have the same positive solution x. Find b.

    (a) $\dfrac{-1+\sqrt{13}}{2}$ (c) $\dfrac{1+\sqrt{13}}{2}$

    (b) $\dfrac{-1+\sqrt{13}}{6}$ (d) $\dfrac{1+\sqrt{13}}{6}$

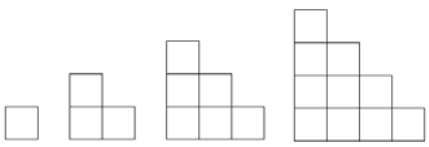
11. If all the words (meaningful or meaningless) obtained from permuting the letters of the word SMART are arranged alphabetically, what is the rank of the word SMART?

    (a) 72nd (b) 79th (c) 80th (d) 78th

12. Evaluate $\sqrt[3]{\sqrt{32}}\sqrt[4]{\sqrt[3]{32}}\sqrt[5]{\sqrt[4]{32}}\cdots\sqrt[10]{\sqrt[9]{32}}$.

    (a) 4 (b) 8 (c) 2 (d) 16

13. The figures below, consisting of unit squares, are the first four in a sequence of "staircases," How many unit squares are there in the 2013th staircase?



    (a) 2014×1007 (b) 2013×1007 (c) 2013×1008 (d) 2013×2007

14. How many factors of 7^{9999} are greater than 1000000?

(a) 9989 (b) 9990 (c) 9991 (d) 9992

15. Let $\overline{AB}$ be a chord of circle C with radius 13. If the shortest distance of $\overline{AB}$ to point C is 5, what is the perimeter of ΔABC?

(a) 30 units (b) 60 units (c) 50 units (d) 25 units

PART II. Each correct answer is worth three points.

1. Find the range of the function $f(x) = \dfrac{4^{x+1} - 3}{4^x + 1}$

(a) $(-3, 1)$ (c) $(-3, 4)$

(b) $(-4, 3)$ (d) $(4, +\infty)$

2. If k consecutive integers sum to -1, find the sum of the largest and smallest term.

(a) -1 (b) 0 (c) 1 (d) k

3. If $2xy + y = 43 + 2x$ for positive integers x, y, find the largest value of $x + y$.

(a) 10 (b) 13 (c) 14 (d) 17

4. Let (a, b) and (c, d) be two points of the circles C_1 and C_2. The circle C_1 is centered at the origin and passes through $P(16, 16)$, while the circle C_2 is centered at P and passes through the origin. Find $a + b + c + d$.

(a) 16 (b) 32 (c) $16\sqrt{2}$ (d) $\dfrac{32}{\sqrt{3}}$

5. Let f be a function that satisfies $f(x + y) = f(x)f(y)$ and $f(xy) = f(x) + f(y)$ for all real numbers x, y. Find $f(\pi^{2013})$.

(a) 2013 (b) 0 (c) 1 (d) π

6. Evaluate $\log_2 \sin(\pi/8) + \log_2 \cos(15\pi/8)$.

(a) 1/2 (b) 0 (c) -1 (d) $-3/2$

7. If the sum of the infinite geometric series $\frac{a}{b} + \frac{a}{b^2} + \frac{a}{b^3} + \cdots$ is 4, then what is the sum of

$$\frac{a}{a+b} + \frac{a}{(a+b)^2} + \frac{a}{(a+b)^3} + \cdots ?$$

(a) 0.8 (b) 1 (c) 1.2 (d) 1.6

8. How many trailing zeros does 126! have when written in decimal notation?

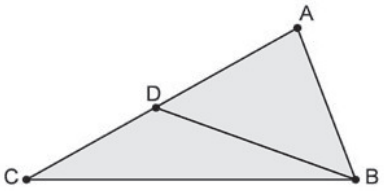
(a) 30 (b) 31 (c) 25 (d) 12

9. It is known that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$. Find the sum

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots .$$

(a) $\frac{\pi^2}{8}$ (b) $\frac{\pi^2}{9}$ (c) $\frac{\pi^2}{10}$ (d) $\frac{\pi^2}{12}$

10. In ΔABC, a point D is on $\overline{AC}$ so that $AB = AD$ so that $m\angle ABC - m\angle ACB = 45°$. Find $m\angle CBD$.



(a) 15° (b) 18° (c) 22.5° (d) 30°

PART III. Each correct answer is worth six points.

1. Let a_n be a sequence such that the average of the first and second terms is 1, the average of the second and third terms is 2, the average of the third and fourth terms is 3, and so on. Find the average of the 1st and 100th terms.

(a) 75 (b) 25 (c) 50 (d) 100

2. Let f be a function such that $f(0) = 1$ and

$$f(2xy - 1) = f(x)f(y) - f(x) - 2y - 1$$

for all x and y. Which of the following is true?

(a) $f(x) \geq 0$ for all real x.
(b) $f(5)$ is a composite number.
(c) $f(7)$ is an even integer.
(d) $f(12)$ is a perfect square.

3. If $\dfrac{a}{a^2+1} = \dfrac{1}{3}$, determine $\dfrac{a^3}{a^6+a^5+a^4+a^3+a^2+a+1}$.

(a) $\dfrac{1}{25}$ (b) $\dfrac{1}{29}$ (c) $\dfrac{1}{33}$ (d) $\dfrac{1}{36}$

4. If $m^3 - 12mn^2 = 40$ and $4n^3 - 3m^2n = 10$, find $m^2 + 4n^2$.

(a) $6\sqrt[3]{2}$ (b) $8\sqrt[3]{2}$ (c) $9\sqrt[3]{2}$ (d) $10\sqrt[3]{2}$

5. Find the minimum value of $2a^8 + 2b^6 + a^4 - b^3 - 2a^2 - 2$, where a and b are real numbers.

(a) $3/8$ (b) $5/8$ (c) $-11/4$ (d) $-11/8$

Answers:

PART 1 (1) A (2) B (3) B (4) C (5) A (6) C (7) D (8) A (9) A (10) A (11) B (12) D (13) B (14) D (15) C

PART 2 (1) C (2) A (3) C (4) B (5) B (6) D (7) A (8) C (9) A (10) C

PART 3 (1) C (2) D (3) B (4) D (5) C