16th Philippine Mathematical Olympiad

National Stage, Oral Phase

25 January 2014

15-Second ROUND

EASY

If $x = \sqrt{2013 - yz}$, $y = \sqrt{2014 - zx}$ and $z = \sqrt{2015 - xy}$, find

$$(x + y)^2 + (y + z)^2 + (z + x)^2.$$

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$$(x+y)^2 + (y+z)^2 + (z+x)^2.$$

Answer: 12084

What is the smallest number of integers that need to be selected from $\{1, 2, \ldots, 50\}$ to guarantee that two of the selected numbers are relatively prime?

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Answer: 26

EASY ROUND
Problem 3

A horse is tied outside a fenced triangular garden at one of the vertices. The triangular fence is equilateral with side length equal to 8 m. If the rope with which the horse is tied is 10 m long, find the area over which the horse can graze outside the fence assuming that the rope and the fence are strong enough to hold the animal.

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Answer: 86π m^2

Let a, b and c be positive numbers such that $a + b + ab = 15$, $b + c + bc = 99$ and $c + a + ca = 399$.
Find

$$a + b + c + bc.$$

EASY ROUND
Problem 4

Let a, b and c be positive numbers such that $a+b+ab=15$, $b+c+bc=99$ and $c+a+ca=399$.
Find

$$a+b+c+bc.$$

Answer: 106

A set S has n elements. There are exactly 57 subsets of S with two or more elements. How many elements does S have?

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Answer: 6

In rectangle $ABCD$, point E is chosen in the interior of AD, and point F is chosen in the interior BC. Let AF and BE meet at G, and CE and DF at H. The following areas are known: $[\triangle AGB] = 9$, $[\triangle BGF] = 16$, $[\triangle CHF] = 11$ and $[\triangle CHD] = 15$. Find the area of quadrilateral $EGFH$.

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Answer: 24

Find the largest value of $4x - x^4 - 1$ as x runs through all real numbers.

Find the largest value of $4x - x^4 - 1$ as x runs through all real numbers.

Answer: 2

Given two distinct points A and B, a point P is randomly and uniformly chosen in the interior of segment AB. Let $r > 0$. Find the probability, in terms of r, that $\frac{AP}{BP} < r$.

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Answer: $\frac{r}{r+1}$

How many triples (x, y, z) of positive integers satisfy the equation $x^{y^z} y^{z^x} z^{x^y} = 3xyz$?

EASY ROUND
Problem 9

How many triples (x, y, z) of positive integers satisfy the equation $x^{y^z} y^{z^x} z^{x^y} = 3xyz$?

Answer: 6

EASY ROUND
Problem 10

The first term of a sequence $\{a_n\}$ is $a_1 = 1$ and for $n \geq 2$, $a_{n+1} = \dfrac{a_n}{1 + ca_n}$ for some constant c. If $a_{12} = \dfrac{1}{2014}$, find c.

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Answer: 183

EASY ROUND
Problem 11

Suppose a, b and c are the roots of $x^3 - 4x + 1 = 0$. Find the numerical value of

$$\frac{a^2bc}{a^3+1} + \frac{ab^2c}{b^3+1} + \frac{abc^2}{c^3+1}.$$

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$$\frac{a^2bc}{a^3+1} + \frac{ab^2c}{b^3+1} + \frac{abc^2}{c^3+1}.$$

Answer: $-\frac{3}{4}$

Find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{\left[\left[\cos\left(\frac{2\pi}{n}\right) + 1\right]\right]}{4^n}$.

EASY ROUND
Problem 12

Find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{\left[\left[\cos\left(\frac{2\pi}{n}\right)+1\right]\right]}{4^n}$.

Answer: $\dfrac{97}{192}$

How many three digit positive integers are there, the sum of whose digits is a perfect cube?

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Answer: 38

Find the greatest possible value of the constant k so that for any 5 positive real numbers a_1, a_2, a_3, a_4, a_5, with sum denoted by S, the following inequality will always be true:

$$(S - a_1)(S - a_2)(S - a_3)(S - a_4)(S - a_5) \geq k(a_1a_2a_3a_4a_5).$$

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Answer: $2^{10} = 1024$

EASY ROUND

Problem 15

Six matchsticks, each 1 inch long, are used to form a pyramid having equilateral triangles for its 4 faces. What is the volume of the pyramid?

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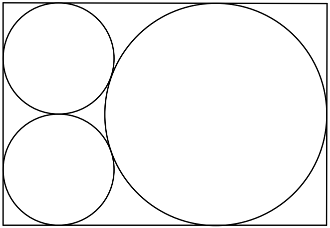
Answer: $\dfrac{\sqrt{2}}{12}\ \text{in}^3$

AVERAGE

AVERAGE ROUND

Problem 1

Two congruent circles and a bigger circle are tangent with each other and the sides of a rectangle, as shown in the figure. If the two congruent circles have common radius r, find the length of the rectangle's longer side.



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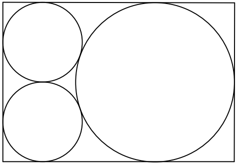

Answer: $(3+2\sqrt{2})r$

AVERAGE ROUND

Problem 2

Find the remainder when $3!^{5!^{7!^{\cdots^{2013!}}}}$ is divided by 11.

AVERAGE ROUND

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Find the remainder when $3!^{5!^{7!^{\cdots^{2013!}}}}$ is divided by 11.

Answer: 1

Let $n \leq m$ be positive integers such that the first n numbers in $\{1, 2, 3 \ldots, m\}$, and the last $m - n$ numbers in the same sequence, have the same sum 3570. Find m.

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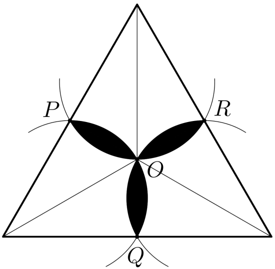
Answer: 119

AVERAGE ROUND
Problem 4

In an equilateral triangle of side length 2, let P, Q and R be the midpoints of the three sides, and O the centroid. Three circles are constructed, each passing through O and two of the midpoints, as shown in the figure. Find the area of the shaded region.



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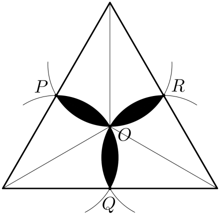

Answer: $\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}$

Suppose that $w + 4x + 9y + 16z = 6$, $4w + 9x + 16y + 25z = 7$, $9w + 16x + 25y + 36z = 12$. Find $w + x + y + z$.

Suppose that $w + 4x + 9y + 16z = 6$, $4w + 9x + 16y + 25z = 7$, $9w + 16x + 25y + 36z = 12$. Find $w + x + y + z$.

Answer: 2

A bag contains 3 red and 3 green balls. Three balls are drawn at random from the bag and replaced with 3 white balls. Afterwards, another 3 balls are drawn at random from the bag. Find the probability that the color of the 3 balls in the second draw are all different.

AVERAGE ROUND

Problem 6

A bag contains 3 red and 3 green balls. Three balls are drawn at random from the bag and replaced with 3 white balls. Afterwards, another 3 balls are drawn at random from the bag. Find the probability that the color of the 3 balls in the second draw are all different.

Answer: $\dfrac{27}{100}$

What is the largest positive integer abcdef that can be formed from the digits $1, 2, 3, 4, 5, 6$, each used exactly once, if abcdef is divisible by 6, abcde is divisible by 5, abcd is divisible by 4, abc is divisible by 3, and ab is divisible by 2?

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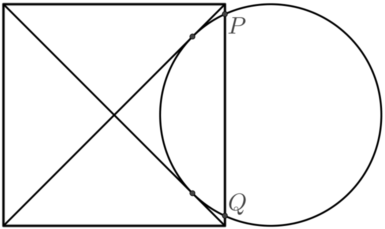
Answer: 321654

Let N be the smallest integer such that the quotient $\dfrac{10!}{N}$ is odd. If x and y are nonnegative numbers such that $2x + y = N$, what is the maximum value of x^2y^2?

Let N be the smallest integer such that the quotient $\frac{10!}{N}$ is odd. If x and y are nonnegative numbers such that $2x + y = N$, what is the maximum value of x^2y^2?

Answer: 2^{26}

A circle with diameter 2 is tangent to the diagonals of a square with side of length 2, as shown the figure. The circle intersects the square at points P and Q. Find the length of segment PQ.



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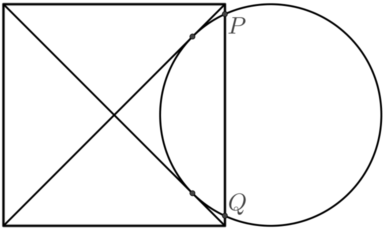

Answer: $\sqrt{8(\sqrt{2}-1)}$

Find the maximum value of the expression

$$\sqrt{(x-4)^2+(x^3-2)^2}-\sqrt{(x-2)^2+(x^3+4)^2}$$

as x runs through all real numbers.

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$$\sqrt{(x-4)^2+(x^3-2)^2}-\sqrt{(x-2)^2+(x^3+4)^2}$$

as x runs through all real numbers.

Answer: $2\sqrt{10}$

DIFFICULT

In parallelogram $PQRS$, $\angle QPS = 45^\circ$, $\angle PQS = 30^\circ$, and $PS = \sqrt{2}$. Find the distance of Q from diagonal PR.

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Answer: $\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

Solve for x:

$$\sqrt{x+\sqrt{3x+6}}+\sqrt{x-\sqrt{3x+6}}=6.$$

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$$\sqrt{x+\sqrt{3x+6}}+\sqrt{x-\sqrt{3x+6}}=6.$$

Answer: 10

Find all $0 \leq \theta \leq 2\pi$ satisfying

$$\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\cos 8\theta}}}=\cos\theta.$$

DIFFICULT ROUND
Problem 3

Find all $0 \leq \theta \leq 2\pi$ satisfying

$$\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{\frac{1}{2} + \frac{1}{2}\cos 8\theta}}} = \cos\theta.$$

Answer: $\theta \in \left[0, \frac{\pi}{8}\right] \cup \left[\frac{7\pi}{8}, 2\pi\right]$

DIFFICULT ROUND

Problem 4

In how many ways can the letters in the word PHILLIP be arranged so that both of the strings PHI or ILL do not appear?

In how many ways can the letters in the word PHILLIP be arranged so that both of the strings PHI or ILL do not appear?

Answer: 522

Solve the equation $2x\left(x - [\![x]\!]\right) = [\![x]\!]^2$.

DIFFICULT ROUND

Problem 5

Solve the equation $2x\left(x - [\![x]\!]\right) = [\![x]\!]^2$.

Answer: $x = 0, \dfrac{\sqrt{3}+1}{2}, \sqrt{3}+1$