13^{th} Philippine Mathematical Olympiad

Area Stage

20 November 2010

Part I. No solution is needed. All answers must be in simplest form. Each correct answer is worth three points.

1. Find the solution set to the equation $(x^2 - 5x + 5)^{x^2-9x+20} = 1$.
2. Suppose $x(x - b - 3) = -2(b + 1)$. Find x.
3. The quotient of the sum and difference of two integers is 3, while the product of their sum and difference is 300. What are the integers?
4. Find the last 2 nonzero digits of 16!
5. Let $f(x)$ be a cubic polynomial. If $f(x)$ is divided by $2x + 3$, the remainder is 4, while if it is divided by $3x + 4$, the remainder is 5. What will be the remainder when $f(x)$ is divided by $6x^2 + 17x + 12$?
6. The operation $*$ satisfies the following properties:
$$x * 0 = 0, \quad x * (y + 1) = x * y + (x - y).$$
Evaluate $2010 * 10$.
7. Find the probability of obtaining two numbers x and y in the interval $[0, 1]$ such that $x^2 - 3xy + 2y^2 > 0$.
8. Find all complex numbers x satisfying $x^3 + x^2 + x + 1 = 0$.
9. Find the range of the function $f(x) = 2^{x^2-4x+1}$.
10. A "fifty percent mirror" is a mirror that reflects half the light shined on it back and passes the other half of the light onward. Now, two "fifty percent mirrors" are placed side by side in parallel and a light is shined from the left of the two mirrors. How much of the light is reflected back to the left of the two mirrors?
11. Find the sum of the coefficients of the polynomial $\cos(2 \arccos(1 - x^2))$.
12. Let $s_1 = 2^{2010}$. For $n > 2$, define
$$s_{n+1} = \begin{cases} \log_{\sqrt{2}} s_n, & s_n > 0 \\ 0, & s_n \leq 0 \end{cases}$$
Find the smallest n such that $s_n \in [4, 6]$.
13. Two students, Lemuel and Christine, each wrote down an arithmetic sequence on a piece of paper. Lemuel wrote down the sequence $2, 9, 16, 23, ...$, while Christine wrote down the sequence $3, 7, 11, 15, ...$ After they have both written out 2010 terms of their respective sequences, how many numbers have they written in common?

14. The line from the origin to the point $(1, \tan 75°)$ intersect the unit circle at P. Find the slope of the tangent line to the circle at P.

15. Let $f(x)$ be a nonzero function whose domain and range is the set of complex numbers. Find all complex numbers x such that $f(x^2) + xf\left(\frac{1}{x^2}\right) = \frac{1}{x}$.

16. Consider addition $\oplus$ and multiplication $\otimes$ modulo 7 of the numbers in $S = \{0, 1, 2, 3, 4, 5, 6\}$. This means that

$$m \oplus n = \text{remainder when } m + n \text{ is divided by } 7$$

$$m \otimes n = \text{remainder when } m \times n \text{ is divided by } 7.$$

Then 1 is the multiplicative identity and each element $a \in S$ has a multiplicative inverse $\frac{1}{a}$. Find the value of $\frac{1}{4} \oplus \left(2 \otimes \frac{1}{3}\right)$ in this number system.

17. Find all real numbers a such that $x^3 + ax^2 - 3x - 2$ has two distinct real zeros.

18. A circle with center C and radius r intersects the square $EFGH$ at H and at M, the midpoint of EF. If C, E and F are collinear and E lies between C and F, what is the area of the region outside the circle and inside the square in terms of r?

19. What is the remainder when $(0! + 1! + 2! + \cdots + 2011!)^2$ is divided by 10?

20. Let $a = 444 \cdots 444$ and $b = 999 \cdots 999$ (both have 2010 digits). What is the $2010th$ digit of the product ab?

Part II. Show the solution to each item. Each complete and correct solution is worth ten points.

1. Sherlock and Mycroft play a game which involves flipping a single fair coin. The coin is flipped repeatedly until one person wins. Sherlock wins if the sequence TTT (tails-tails-tails) shows up first while Mycroft wins if the sequence HTT(heads-tails-tails) shows up first. Who among the two has a higher probability of winning?

2. Denote by a, b and c the sides of a triangle, opposite the angles α, β and γ, respectively. If α is sixty degrees, show that $a^2 = \frac{a^3 + b^3 + c^3}{a + b + c}$.

3. Show that $\sqrt[n]{2} - 1 \leq \sqrt{\frac{2}{n(n-1)}}$ for all positive integers $n \geq 2$.

1. $\{1,2,3,4,5\}$
2. $x = b+1,\ 2$
3. $(20,10),\ (-20,-10)$
4. 88
5. $6x+13$
6. 20,055
7. $\frac{3}{4}$
8. $x = -1, i, -i$
9. $[\frac{1}{8}, \infty)$
10. $\frac{2}{3}$
11. $-4+1+2=-1$
12. 6
13. 287
14. $-\dfrac{1}{\tan(30^\circ+45^\circ)} = -2+\sqrt{3}$
15. This item was scrapped.
16. 5
17. $a = 0$
18. $r^2\left(\dfrac{22}{25} - \dfrac{\tan^{-1}(4/3)}{2}\right)$
19. 6
20. 3

Part II

1. Sherlock has probability 1/8 of winning while Mycroft has probability greater than $\frac{1}{8}$. The event "*Sherlock wins*" is just the set $\{TTT\}$ so that $P(\{TTT\}) = \frac{1}{8}$ while the event "*Mycroft wins*" is the set $M = \{HTT, HHTT, THTT, HHHTT, TTHTT, HTHTT, THHTT, \cdots\}$ and $P(M) > \frac{1}{8}$.

2. $a^2 = b^2 + c^2 - 2bc(\frac{1}{2})$ by cosine law, and $b^3 + c^3 = (b+c)(b^2 - bc + c^2) = (b+c)a^2$. Add a^3 to both sides and move terms to get the desired equation. That is, $a^3 + b^3 + c^3 = (b+c)a^2 + a^3 = (a+b+c)a^2$ and the desired equality follows.

3. Let $x_n = \sqrt[n]{2} - 1 \geq 0$. Then $2 = (1+x_n)^n \geq 1 + nx_n + \dfrac{n(n-1)}{2}x_n^2 \geq 1 + \dfrac{n(n-1)}{2}x_n^2$. Thus, $\dfrac{n(n-1)}{2}x_n^2 \leq 2-1$ and the desired inequality follows.